CL-shellable posets with no EL-shellings

Tiansi Li

Washington University in St. Louis

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Poset

A partially ordered set (also called poset) is a set P equipped with a partial order, which we usually denote with $\leq$

- Reflexivity: $\forall x \in P, x \leq x$
- Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$
- Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z$

Hasse Diagrams

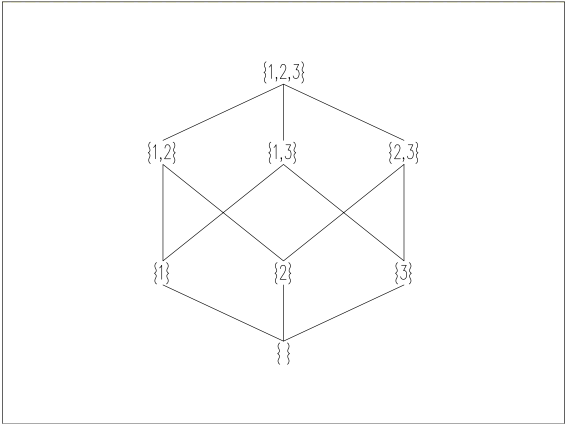


Figure: Boolean lattice on [3]

Hasse Diagrams

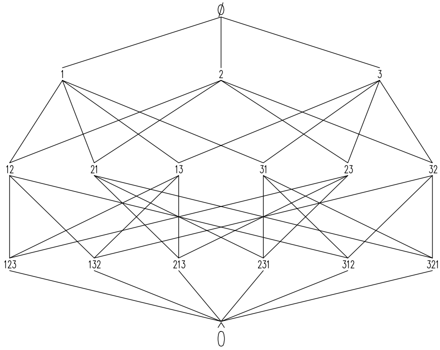
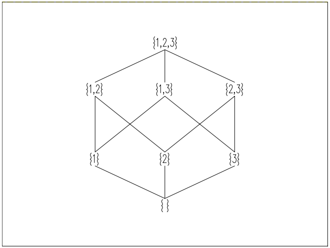
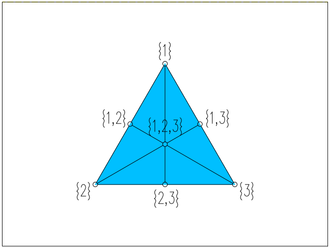


Figure: Injective words poset on [3]

Order Complex

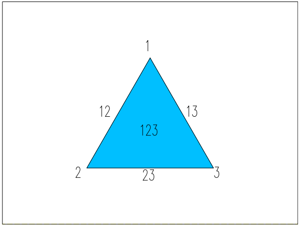
Let P be a finite poset. The order complex of P, denoted by ΔP, is the simplicial complex with $V(\Delta P) = P$ and $F = \{v_1, v_2, \ldots, v_k\} \in \Delta P$ if $v_1 < v_2 < \cdots < v_k$ in P.

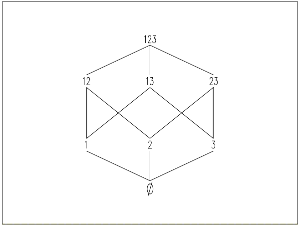




Face Poset

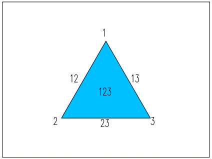
Let Δ be a finite simplicial complex. The face poset of Δ, denoted by $P(\Delta)$, is the poset whose elements are faces of Δ and partially ordered by face inclusion.

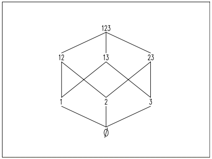


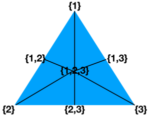


Barycentric Subdivision

Let Δ be a finite simplicial complex, then the order complex of its face poset is the barycentric subdivision of Δ.







Shellable Complex

Let Δ be a finite simplicial complex. Δ is said to be shellable if its facets can be ordered as $F_1, F_2, \ldots, F_k$ such that for each i, $F_i \cap (\cup_{j=1}^{i-1} F_j)$ is pure of codimension 1 in F_i.

We call a poset P shellable if ΔP is shellable.

Shellable Complex

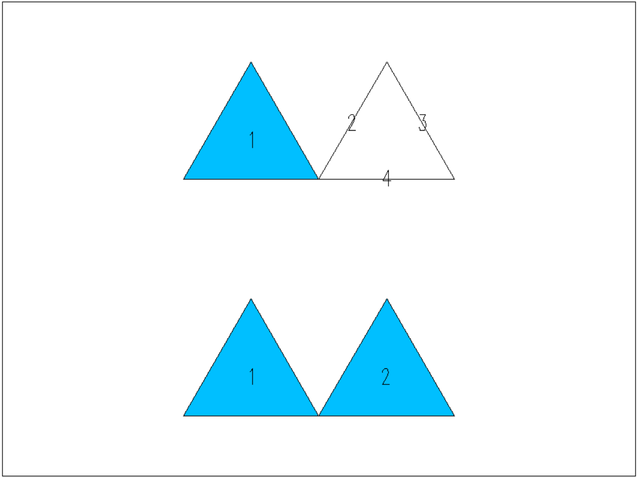


Figure: A shellable complex and a non-shellable complex

Shellable Complex

Theorem(Björner-Wachs 96)

A shellable simplicial complex has the homotopy type of a wedge of spheres (in varying dimensions), where for each i, the number of i-spheres is the number of i-facets whose entire boundary is contained in the union of the earlier facets. Such facets are usually called homology facets.

Corollary

If Δ is shellable then for all i, $\tilde{H}_i(\Delta, \mathbb{Z}) \cong \tilde{H}^i(\Delta, \mathbb{Z}) \cong \mathbb{Z}^{r_i}$, where r_i is the number of homology i-facets in Δ.

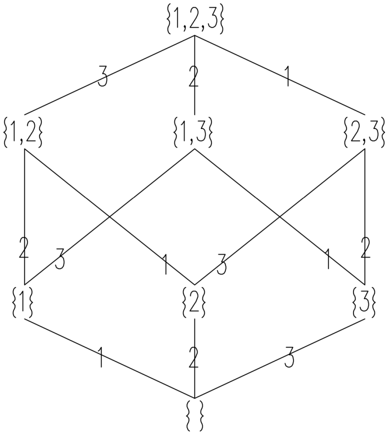
Edge Labeling

An edge labeling of a bounded poset P is a map $\lambda : E(P) \to \Lambda$, where $E(P)$ is the set of edges of the Hasse diagram of P , i.e., the covering relations $x \lessdot y$ of P , and Λ is some poset (usually the integers $\mathbb{Z}$ with its natural total order relation).

Given an edge labeling $\lambda : E(P) \to \Lambda$, one can associate a word $\lambda(c) = \lambda(\hat{0}, x_1)\lambda(x_1, x_2), \ldots, \lambda(x_t, \hat{1})$ with each maximal chain $c = (\hat{0} \lessdot x_1 \lessdot \cdots \lessdot x_t \lessdot \hat{1})$. We say that c is weakly increasing if the associated word $\lambda(c)$ is weakly increasing. That is, c is weakly increasing if $\lambda(\hat{0}, x_1) \leq \lambda(x_1, x_2), \ldots, \leq \lambda(x_t, \hat{1})$.

EL-Labeling

Let P be a bounded poset. An edge-lexicographical labeling (EL-labeling, for short) of P is an edge labeling such that in each closed interval $[x, y]$ of P, there is a unique weakly increasing maximal chain, which lexicographically precedes all other maximal chains of $[x, y]$.



EL-Labeling

Theorem(Björner 80, Björner-Wachs 96)

Suppose P is a bounded poset with an EL-labeling. Then the lexicographic order of the maximal chains of P is a shelling of $\Delta(P)$. Moreover, the corresponding order of the maximal chains of $\bar{P}$ is a shelling of$\Delta(\bar{P})$.

Corollary(Björner-Wachs 96)

Suppose P is a poset for which $\hat{P}$ admits an EL-labeling. Then P has the homotopy type of a wedge of spheres, where the number of i-spheres is the number of decreasing maximal $(i+2)$-chains of $\hat{P}$. The decreasing maximal $(i+2)$-chains, with $\hat{0}$ and $\hat{1}$ removed, form a basis for cohomology $\tilde{H}^i(P, \mathbb{Z})$).

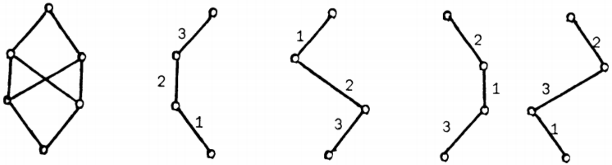
Chain-edge Labeling

For a bounded poset P, let $ME(P)$ be the set of pairs $(c, x \lessdot y)$ consisting of a maximal chain c and an edge $x \lessdot y$ along that chain. A chain-edge labeling of P is a map $\lambda : ME(P) \to \Lambda$, where Λ is some poset, satisfying:
If two maximal chains coincide
along their bottom d edges, then their labels also coincide along these edges.

It follows from the definition that each maximal chain r of $[\hat{0}, x]$ determines a unique restriction of λ to $ME([x, y])$. This enables one to talk about increasing and decreasing maximal chains and lexicographic order of maximal chains in the rooted interval $[x, y]_r$.

CL-Labeling

Let P be a bounded poset. A chain-lexicographic labeling (CL-labeling, for short) of P is a chain-edge labeling such that in each closed rooted interval $[x, y]_r$ of P, there is a unique weakly increasing maximal chain, which lexicographically precedes all other maximal chains of $[x, y]_r$. A poset that admits a CL-labeling is said to be CL-shellable.



CL-Shellable

Theorem(Björner-Wachs 96)

Suppose P is a bounded poset with an CL-labeling. Then the lexicographic order of the maximal chains of P is a shelling of $\Delta(P)$. Moreover, the corresponding order of the maximal chains of $\bar{P}$ is a shelling of$\Delta(\bar{P})$.

Corollary

EL-shellability $\Rightarrow$ CL-shellability $\Rightarrow$ Shellability.

Question

Question(Björner-Wachs 83)

Is every CL-shellable poset EL-shellable?

Recursive Atom Ordering

A bounded poset P is said to admit a recursive atom ordering if its length $l(P)$ is 1, or if $l(P) > 1$ and there is an ordering $a_1, a_2, ..., a_t$ of the atoms of P that satisfies:

- For all $j = 1, 2, \ldots, t$ the interval $[a_j, \hat{1}]$ admits a recursive atom ordering in which the atoms of $[a_j, \hat{1}]$ that belong to $[a_i, \hat{1}]$ for some $i < j$ come first.
- For all $i < j$,if $a_i, a_j < y$ then there is a $k < j$ and an atom z of $[a_j, \hat{1}]$ such that $a_k < z \leq y$.

A recursive coatom ordering is a recursive atom ordering of the dual poset P^*.

Example

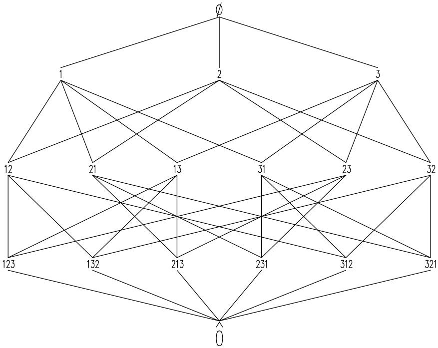


Figure: Injective words poset on [3]

Recursive Atom Ordering

Theorem(Björner-Wachs 83, 96)

A bounded poset P is CL-shellable if and only if P admits a recursive atom ordering.

Theorem(Björner-Wachs 83)

An ordering of the facets of a simplicial complex Δ is a shelling if and only if the ordering is a recursive coatom ordering of the face poset $P(\Delta)$.

Recursive Atom Ordering

Theorem(Li 19)

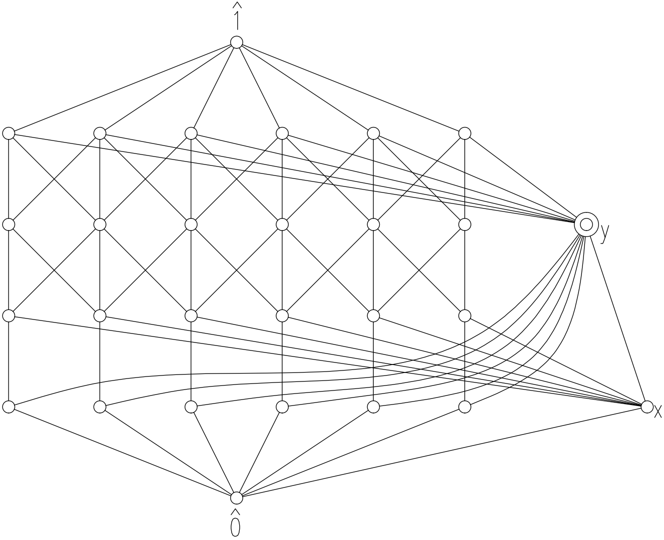
A bounded poset P is EL-shellable if and only if P admits a recursive atom ordering such that, for each pair (x, r), the atom orderings of $[x, \hat{1}]_r$ are the same for all r.

Question

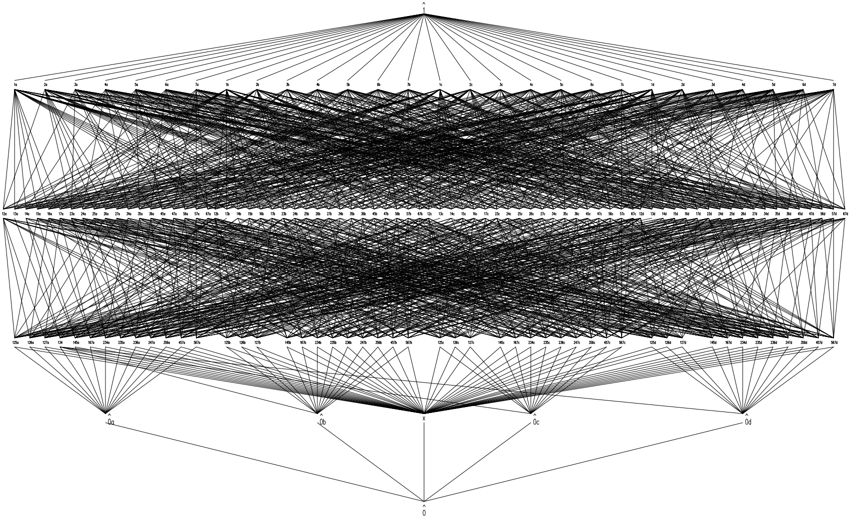
Question(Björner-Wachs 83)

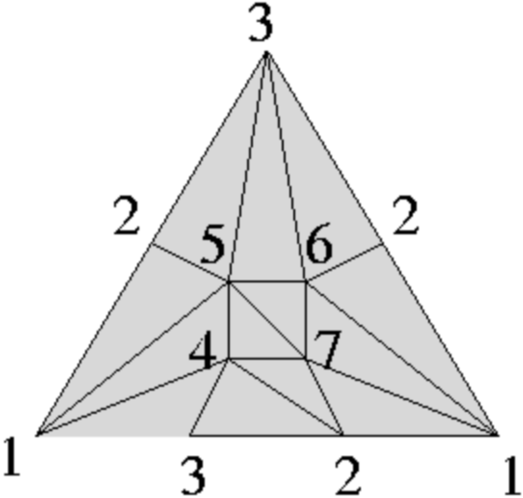
Is every CL-shellable poset EL-shellable?

Ungraded Example



Graded Example

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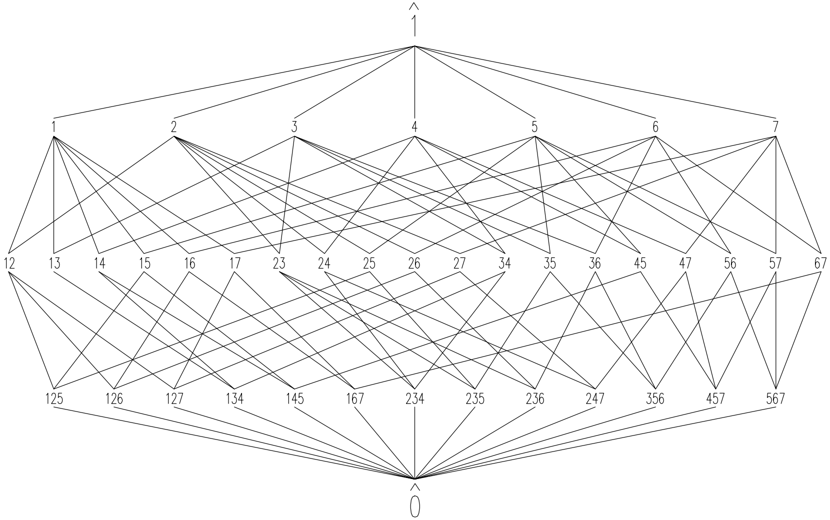
Dual of $P(H)$



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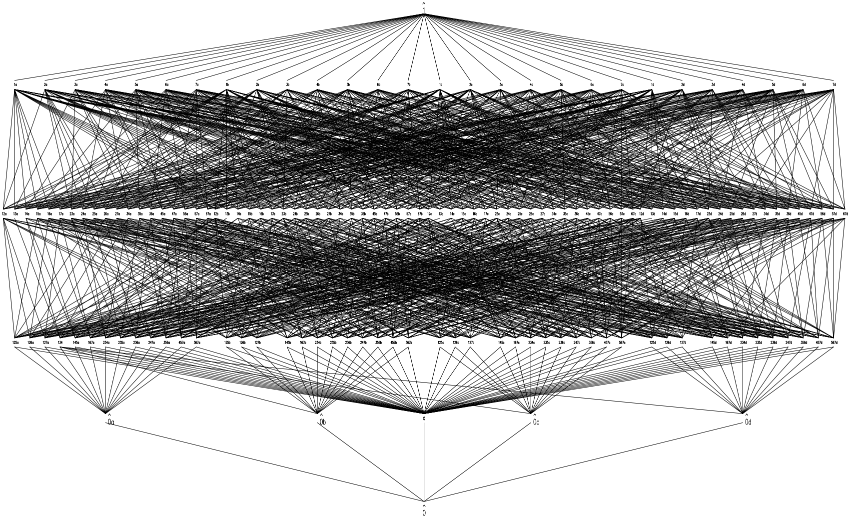
Graded Example



Open Questions

Question

Is there a complete characterization of CL-shellable posets that are not EL-shellable?

Thank you!