The Integer Matrix All-minors Matrix-tree Theorem via Oriented Hypergraphs

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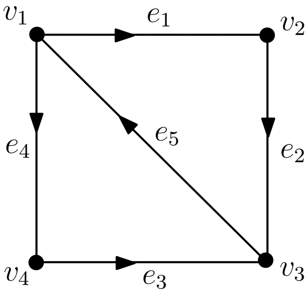
Introduction

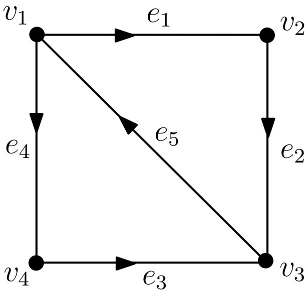
Incidence Matrix - Graph



$$\mathrm{H}_G = \begin{bmatrix} -1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

- $V \times E$ matrix with values $+1$ if the edge enters a vertex and -1 if it exits a vertex.

Incidence Matrix - Graph



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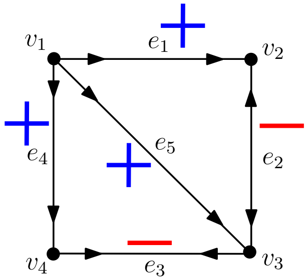
- $V \times E$ matrix with values $+1$ if the edge enters a vertex and -1 if it exits a vertex.
- Edge e_1 exits v_1 and enters v_2

Incidence Matrix - Signed Graph



$$H_G = \begin{bmatrix} -1 & 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$$

- $V \times E$ matrix with values $+1$ if the incidence enters a vertex and -1 if it exits a vertex.

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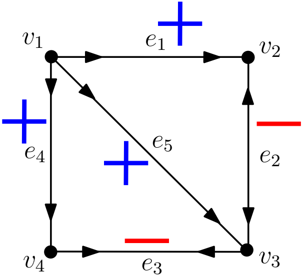

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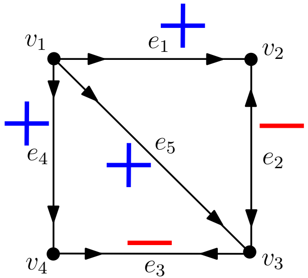

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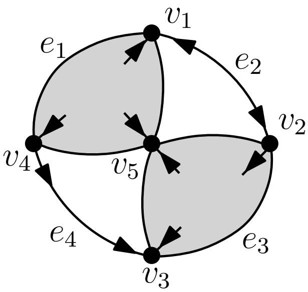
- $V \times E$ matrix with values $+1$ if the incidence enters a vertex and -1 if it exits a vertex.
- Edge e_1 has both incidences oriented to agree with the previous graph.
- Edge e_2 has both incidences entering v_2 and v_3 (extroverted).

Incidence Matrix - Oriented Hypergraph



$$\mathrm{H}_G = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & \mathbf{-1} & 0 \\ 0 & 0 & \mathbf{1} & 1 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & \mathbf{1} & 0 \end{bmatrix}$$

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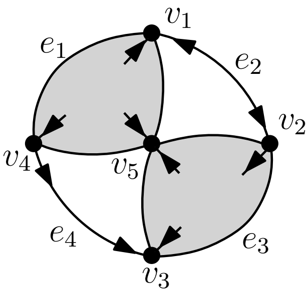

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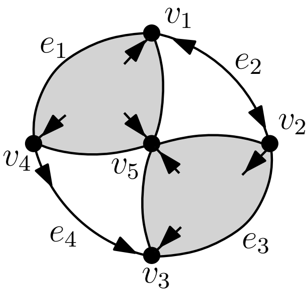
- Edge e_3 has two compatible pairs and one extroverted pair of incidences.

Incidence Matrix - Oriented Hypergraph



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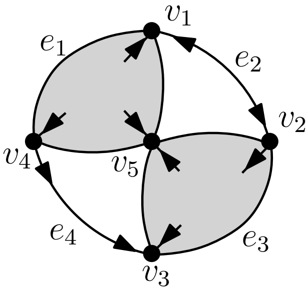
- $V \times E$ matrix with values $+1$ if the <u>incidence</u> enters a vertex and -1 if it exits a vertex.
- Edge e_3 has two compatible pairs and one extroverted pair of incidences.
- No edge of size greater than 2 can have all incidence pairs compatible.

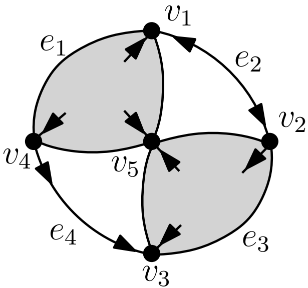
Adjacency Matrix



$$A_G = \begin{bmatrix} 0 & \color{red}{-1} & 0 & -1 & -1 \\ -1 & 0 & \color{purple}{+1} & 0 & +1 \\ 0 & +1 & 0 & \color{teal}{+1} & -1 \\ \color{blue}{-1} & 0 & +1 & 0 & -1 \\ -1 & +1 & -1 & -1 & 0 \end{bmatrix}$$

- Entries are signed by local adjacencies. (Introverted/Extroverted = negative)

Adjacency Matrix



$$A_G = \begin{bmatrix} 0 & -1 & 0 & -1 & -1 \\ -1 & 0 & +1 & 0 & +1 \\ 0 & +1 & 0 & +1 & -1 \\ -1 & 0 & +1 & 0 & -1 \\ -1 & +1 & -1 & -1 & 0 \end{bmatrix}$$

- Entries are signed by local adjacencies. (Introverted/Extroverted = negative)
- The sign of the circle from (v_1, v_2, v_3, v_4) is the product of the adjacency signs.

Sachs' Theorem

Theorem (Sachs' Theorem)

For a graph G the characteristic polynomial is

$$\chi_G(\mathbf{A}, x) = \sum_{k=1}^{|V(G)|} \left(\sum_{U \in \mathscr{U}_k} (-1)^{p(U)} (2)^{c(U)} \right) x^k.$$

Where $\mathscr{U}_k$ is the set of all cycle-covers avoiding k vertices.

- Each cycle-cover is weighted by -1 for each connected component and 2 for each cycle.

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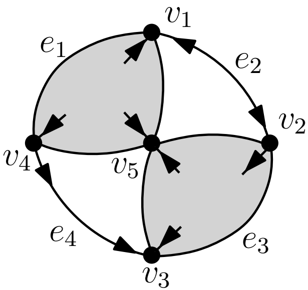
- We obtain a generalization to oriented hypergraphs via the finest possible sum.

Laplacian Matrix

Definition

Laplacian Matrix: $L_G := D_G - A_G = \mathrm{H}_G \mathrm{H}_G^T$



$$L_G = \begin{bmatrix} 2 & 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 & 1 \\ 1 & 0 & -1 & 2 & 1 \\ 1 & -1 & 1 & 1 & 2 \end{bmatrix}$$

- The *degree* of a vertex is the number of incidences at that vertex.

Graphic Matrix-Tree Theorem

Theorem (Matrix-Tree Theorem)

If v is a vertex of a graph G with Laplacian matrix $\mathbf{L}(G)$ then

$$\det\left(\mathbf{L}_v(G)\right) = \sum_T \prod_{e \in E(T)} wt(e)$$

Where the sum is over all spanning trees T, rooted at v, and $wt(e)$ is the weight of edge e.

- If each edge is weighted 1 this simply counts the number of spanning trees of G.

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Signed-Graphic Matrix-Tree Theorem

Theorem (Chaiken's All Minors Matrix-tree Theorem (Chaiken 1982))

Let G be a signed graph with Laplacian matrix $\mathbf{L}$. For $U, W \subseteq V$ with $|U| = |W|$, let $\mathbf{L}_{U,W}$ be (U, W) minor of $\mathbf{L}$ then

$$\det(\mathbf{L}_{U,W}) = \epsilon(\bar{U}, V)\epsilon(\bar{W}, V)\sum_{F}\epsilon(\pi^*)(-1)^{np(F)}4^{nc(F)}a_F$$

Where the sum is over all edge sets F, subset of E, such that

1. *F contains $|U|$ components that are trees.*
2. *Each tree from 1 contains exactly one vertex from U and one vertex from W.*
3. *Each tree from 1 is rooted at its vertex in U and contains exactly one vertex of W. This defines a linking $\pi^* : W \to U$. $\epsilon(\pi^*)$ is negative one to the number of inversions of π^*, and $np(F)$ is the number of negative paths in π^*.*
4. *Each of the remaining components of F contains exclusively a backstep or exactly one negative circle. $nc(F)$ is the number of negative circles.*
5. $\epsilon(\bar{U}, V) = (-1)^{|\{(i,j)|i<j, i\in U, j\in\bar{U}\}|}$

Motivation and Examples

Weak Walks and Path Embeddings

Definition

A *directed weak walk of G* is the image of an incidence-preserving map of a directed path into G.

Definition

A *directed adjacency of G* is a map of $\overrightarrow{P}_1$ into G that is incidence-monic.

Definition

A *backstep of G* is a non-incidence-monic map of $\overrightarrow{P}_1$ into G.

A Unifying Theorem - Weak Walk Theorem

Theorem (Reff & Rusnak, 2012)

The ij-entry of the oriented hypergraphic adjacency matrix is the number of walks of length 1 *from* v_i *to* v_j*.*

Theorem (Reff & Rusnak, 2012)

The ij-entry of the oriented hypergraphic Laplacian matrix is the number of <u>*weak*</u> *walks of length* 1 *from* v_i *to* v_j*.*

- Backsteps correspond to degree.

A Unifying Theorem - Weak Walk Theorem

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The ij-entry of the oriented hypergraphic adjacency matrix is the number of walks of length 1 from v_i to v_j.

Theorem (Reff & Rusnak, 2012)

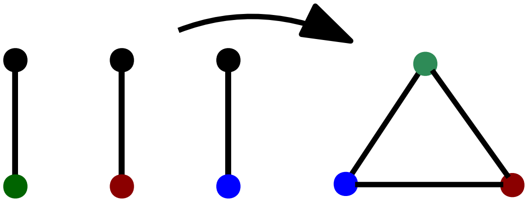
The ij-entry of the oriented hypergraphic Laplacian matrix is the number of weak walks of length 1 from v_i to v_j.

- Backsteps correspond to degree.
- The only difference between **A** and **L** is incidence-monic-ness.

Contributors as Permutation Clones

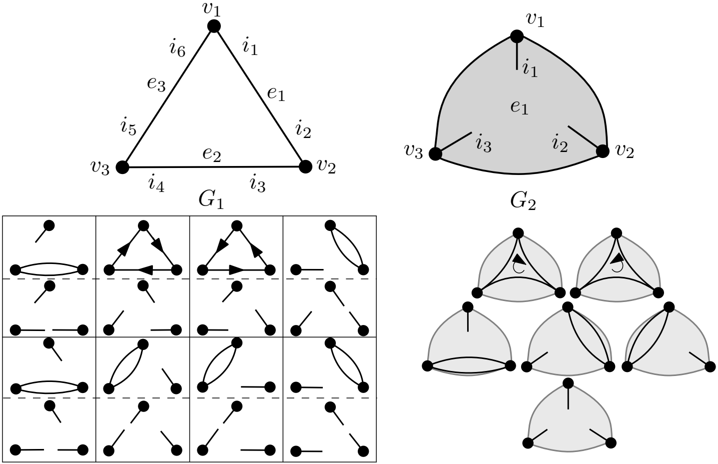
Definition (Contributor)

A *contributor of G* is an incidence preserving map from a disjoint union of $\overrightarrow{P}_1$'s into G defined by $c : \coprod_{v \in V} \overrightarrow{P}_1 \to G$ such that $c(t_v) = v$ and $\{c(h_v) \mid v \in V\} = V$.



- A *strong contributor* is a contributor with no backsteps.

Contributors of K_3 versus E_3



- Both have two strong contributors. (Sachs-figures)

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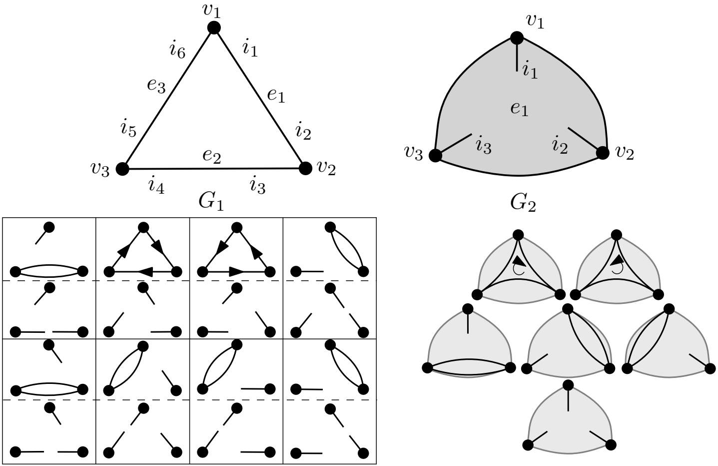


- Both have two strong contributors. (Sachs-figures)
- K_3 has **8** identity clones. Hence, 8 activation classes.

Contributor Sets

Definition

Let $\mathcal{C}(G; \mathbf{u}, \mathbf{w})$ be the set of contributors in G where $c(u_i) = w_i$.

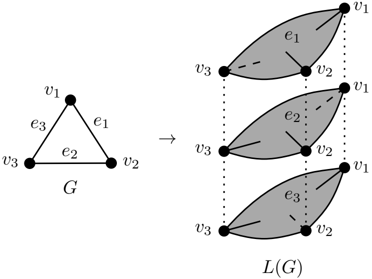
Definition

Let $\widehat{\mathcal{C}}(G; \mathbf{u}, \mathbf{w})$ be the set obtained by removing the $\mathbf{u} \to \mathbf{w}$ mappings from $\mathcal{C}(G; \mathbf{u}, \mathbf{w})$

- $\mathcal{S}(G; \mathbf{u}, \mathbf{w})$ and $\widehat{\mathcal{S}}(G; \mathbf{u}, \mathbf{w})$ will be used to denote the set of *strong* contributors.

Definition

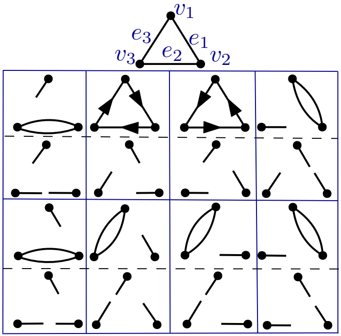
Given an incidence hypergraph G, define the *loading* of G as the incidence hypergraph $L(G)$ that contains G and has an incidence for every (v, e) pair that was incidence-free.



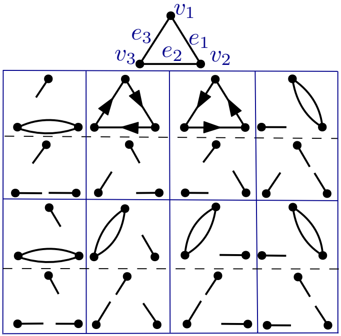
Lemma (Grilliette, R., Rusnak; submitted)

The loading of G is the injective envelope in the category of incidence hypergraphs.

Characteristic Polynomial



Characteristic Polynomial



- $\det(x\mathbf{I} - \mathbf{L}) = x^3 - 6x^2 + 9x$. The constant is 0 as contributors are cancellative within each activation class.

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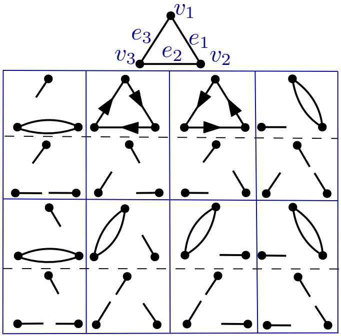
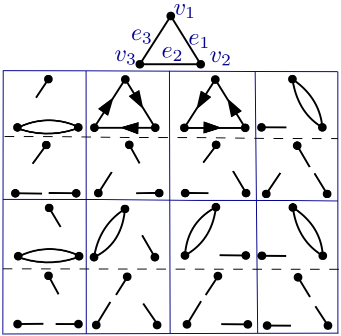
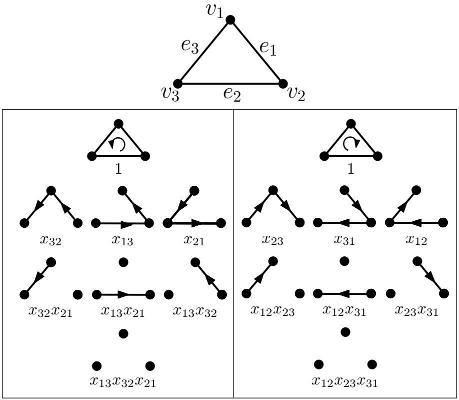


- $\det(x\mathbf{I} - \mathbf{L}) = x^3 - 6x^2 + 9x$. The constant is 0 as contributors are cancellative within each activation class.
- $\mathrm{perm}(x\mathbf{I} - \mathbf{A}) = x^3 + 3x - 2$. The constant is -2 as there are two strong contributors that have been decoupled from their identity.

Expanding to All-minors



$$\mathrm{perm}(\mathbf{X} - \mathbf{A}) = \mathrm{perm}\begin{bmatrix} x_{11} & x_{12} - 1 & x_{13} - 1 \\ x_{21} - 1 & x_{22} & x_{23} - 1 \\ x_{31} - 1 & x_{32} - 1 & x_{33} \end{bmatrix}$$


v_1
e_3
e_1
v_3
e_2
v_2
1
1
x_{32}
x_{13}
x_{21}
x_{23}
x_{31}
x_{12}
$x_{32}x_{21}$
$x_{13}x_{21}$
$x_{13}x_{32}$
$x_{12}x_{23}$
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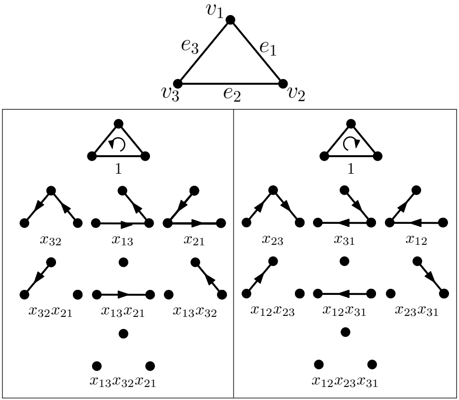
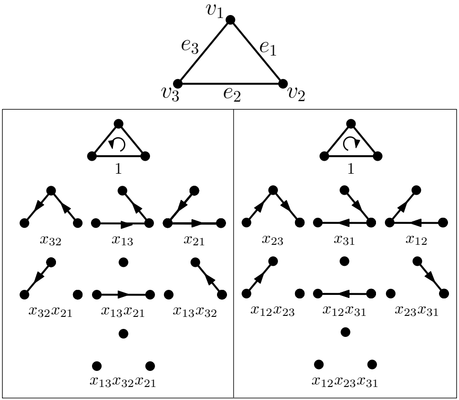
$x_{13}x_{32}x_{21}$
$x_{12}x_{23}x_{31}$



- The constant term will still be produced by the two 3-cycle strong contributors.



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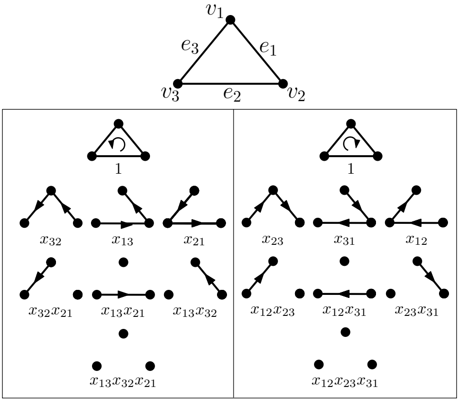
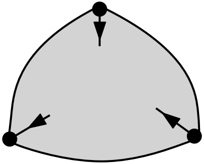


- The constant term will still be produced by the two 3-cycle strong contributors.
- The subcontributors also contribute additional monomials shown.
- $\mathrm{perm}(\mathbf{X} - \mathbf{A})$ is the alternating sum of these monomials.



$$\chi^D(\mathbf{L}_G, \mathbf{x})$$

$$= \det(\mathbf{X} - \mathbf{L}_G) = \det\begin{bmatrix} x_{11}-1 & x_{12}-1 & x_{13}+1 \\ x_{21}-1 & x_{22}-1 & x_{23}+1 \\ x_{31}+1 & x_{32}+1 & x_{33}-1 \end{bmatrix}$$

$$= x_{11}x_{22}x_{33} - x_{11}x_{23}x_{32} - x_{13}x_{22}x_{31} - x_{12}x_{21}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32}$$

$$- x_{11}x_{22} - x_{11}x_{23} - x_{11}x_{32} - x_{11}x_{33} - x_{13}x_{22} - x_{22}x_{31} - x_{22}x_{33} - x_{23}x_{31} - x_{13}x_{32}$$

$$+ x_{12}x_{21} + x_{13}x_{21} + x_{12}x_{23} + x_{12}x_{31} + x_{13}x_{31} + x_{21}x_{32} + x_{23}x_{32} + x_{12}x_{33} + x_{12}x_{33}$$

Note the constant and linear terms all have coefficient zero.

Main Theorems

Theorem (Grilliette, R., Rusnak; submitted)

Let G be an oriented hypergraph with adjacency matrix $\mathbf{A}_G$ and Laplacian matrix $\mathbf{L}_G$, then

1. $$\chi^P(\mathbf{A}_G,\mathbf{x}) = \sum_{[\mathbf{u},\mathbf{w}]} \left(\sum_{\substack{s\in\widehat{\mathcal{S}}(L^0(G);\mathbf{u},\mathbf{w}) \\ \text{sgn}(s)\neq 0}} (-1)^{oc(s)+nc(s)} \right) \prod_i x_{u_i,w_i},$$
2. $$\chi^D(\mathbf{A}_G,\mathbf{x}) = \sum_{[\mathbf{u},\mathbf{w}]} \left(\sum_{\substack{s\in\widehat{\mathcal{S}}(L^0(G);\mathbf{u},\mathbf{w}) \\ \text{sgn}(s)\neq 0}} (-1)^{ec(\check{s})+oc(s)+nc(s)} \right) \prod_i x_{u_i,w_i},$$
3. $$\chi^P(\mathbf{L}_G,\mathbf{x}) = \sum_{[\mathbf{u},\mathbf{w}]} \left(\sum_{\substack{c\in\widehat{\mathcal{C}}(L^0(G);\mathbf{u},\mathbf{w}) \\ \text{sgn}(c)\neq 0}} (-1)^{nc(c)+bs(c)} \right) \prod_i x_{u_i,w_i},$$
4. $$\chi^D(\mathbf{L}_G,\mathbf{x}) = \sum_{[\mathbf{u},\mathbf{w}]} \left(\sum_{\substack{c\in\widehat{\mathcal{C}}(L^0(G);\mathbf{u},\mathbf{w}) \\ \text{sgn}(c)\neq 0}} (-1)^{ec(\check{c})+nc(c)+bs(c)} \right) \prod_i x_{u_i,w_i}.$$

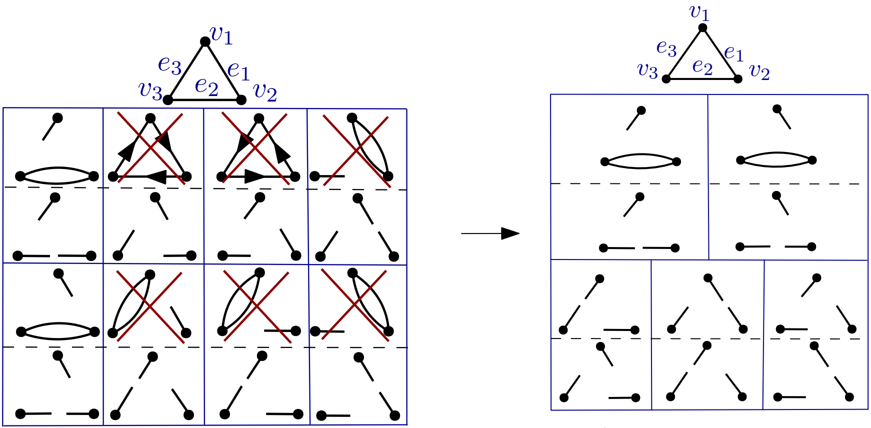
Arborescenes

Definition

Let $\mathcal{A}(\mathbf{u}; \mathbf{w}; G)$ denote the $(\mathbf{u}, \mathbf{w})$-equivalent elements in activation class $\mathcal{A}$.

Definition

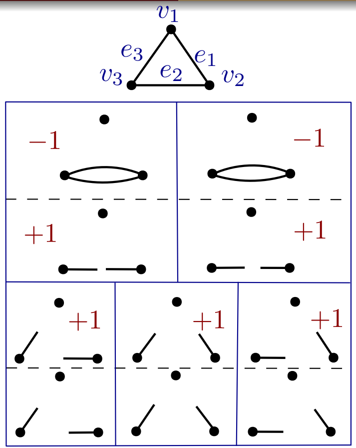
Let $\hat{\mathcal{A}}(\mathbf{u}; \mathbf{w}; G)$ be the elements of $\mathcal{A}(\mathbf{u}; \mathbf{w}; G)$ with the adjacency or backstep from u_i to w_i is removed for each i.



$(1,1)$-contributors of G

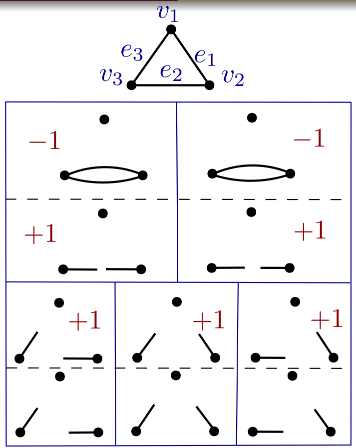
Contributor with x_{11}

- For each activation class take only those contributors where $1 \to 1$.



$$\chi^D(L_G, x_{11}) = 3x_{11}$$

- Non-single-element activation classes are still cancellative.



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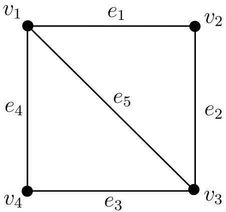
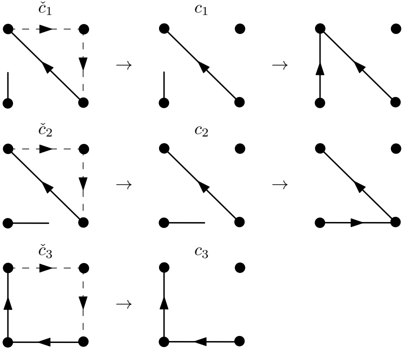
- Non-single-element activation classes are still cancellative.
- The single-element activation classes are unpacking-equivalent to spanning trees.

Arborescenes

The three $[(1,2),(2,3)]$-equivalent contributors, their reduced subcontributor in G with linking, and the unpacked inward arborescence rooted at v_1.





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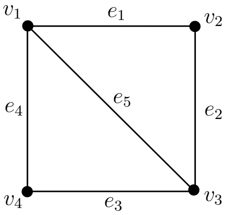
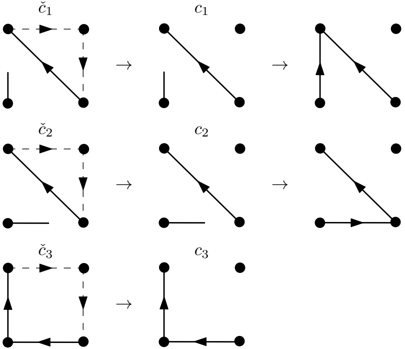
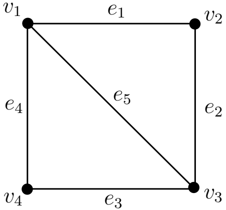


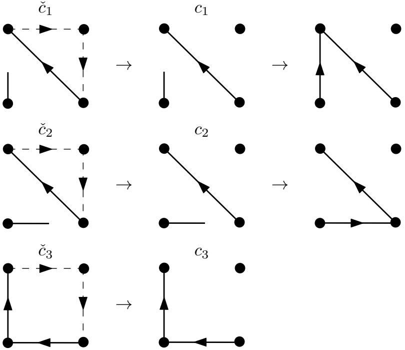


- The coefficient of $x_{12}x_{23}$ in $\chi^D(\mathbf{L}_G, \mathbf{x})$ is -3.

Arborescenes

The three $[(1,2),(2,3)]$-equivalent contributors, their reduced subcontributor in G with linking, and the unpacked inward arborescence rooted at v_1.





- The coefficient of $x_{12}x_{23}$ in $\chi^D(\mathbf{L}_G, \mathbf{x})$ is -3.
- The coefficient of $x_{12}x_{23}$ in $\chi^P(\mathbf{L}_G, \mathbf{x})$ is -1.

Arborescenes

Theorem (Grilliette, R., Rusnak; submitted)

In a bidirected graph G the set of all elements in single-element $\hat{\mathcal{A}}_{\neq 0}(\mathbf{u};\mathbf{w};L(G))$ is unpacking equivalent to k-arborescences. Moreover, the i^{th} component in the arborescence has sink u_i, and the vertices of each component are determined by the linking induced by c^{-1} between all $u_i \in U \cap \overline{W} \to \overline{U}$ or unpack into a vertex of a linking component.

Thanks