On weighted modulo orientation of graphs

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(Joint work with Ping Li, Miaomiao Han, Jiaao Li and Hong-Jian Lai)

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- $G = (V, E)$ is a graph.
- D: Orientation of G.
- $E_D^+(v)$: the set of all arcs directed out from v.
- $E_D^-(v)$: the set of all arcs directed into v.

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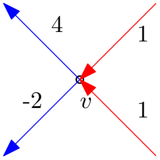
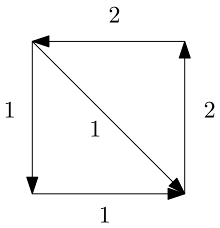
- Nowhere-zero k-flow: the pair (D, f) with $f : E(G) \to \{\pm 1, \pm 2, \dots, \pm (k-1)\}$ such that, for each vertex v,

$$\sum_{e \in E_D^+(v)} f(e) = \sum_{e \in E_D^-(v)} f(e).$$





Nowhere zero 3-flow

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3-Flow Conjecture, 1972

Every 4-edge-connected graph admits a Nowhere-zero 3-flow.

4-Flow Conjecture, 1966

Every bridgeless graph without Peterson-minor admits a Nowhere-zero 4-flow.

5-Flow Conjecture, 1954

Every bridgeless graph admits a Nowhere-zero 5-flow.

- the out-degree of v: $d_D^+(v) = |E_D^+(v)|$
- the in-degree of v: $d_D^-(v) = |E_D^-(v)|$
- Modulo k-orientation D: there is an orientation D such that, for each vertex v,

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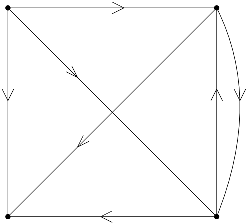

Figure: Modulo 3-orientation

Esperet, De Verclos et al. [1] defined a modulo k f-weighted b-orientation of a graph G.

[1] L. Esperet, R. J. De Verclos, T. N. Le and S. Thomassé, Additive Bases and Flows in Graphs, SIAM J. Discrete Math., 32(1)(2018), 534–542.

Esperet, De Verclos et al. [1] defined a modulo k f-weighted b-orientation of a graph G.

- $(f, b; k)$-orientation: Given any $f : E(G) \to \mathbb{Z}_k - \{0\}$ and any $b : V(G) \to \mathbb{Z}_k$ with $\sum b(v) \equiv 0 \pmod{k}$, there exists D such that for each vertex v:

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- $(f, b; k)$-orientation $\Rightarrow$ Modulo k-orientation.

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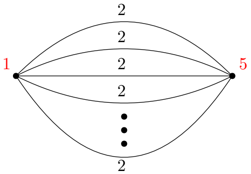

Figure: No $(f, b; 6)$-orientation for large edge connectivity.

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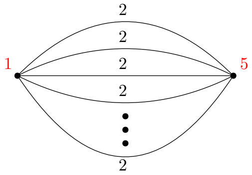

Figure: No $(f, b; 6)$-orientation for large edge connectivity.

Theorem (Esperet, De Verclos, Le and Thomassé, [1])

Let $p \geq 3$ be a prime number and $\kappa'(G) \geq (6p^2 - 14p + 8)$. Then G has an $(f, b; p)$-orientation.

[1] L. Esperet, R. J. De Verclos, T. N. Le and S. Thomassé, Additive Bases and Flows in Graphs, SIAM J. Discrete Math., 32(1)(2018), 534–542.

$\mathcal{O}_p = \{G : G$ is a connected graph and G admits an $(f, b; p)$-orientation$\}$.

Lemma ([2],2019+)

Let G be a connected graph. Then each of following holds.
(i) $K_1 \in \mathcal{O}_p$.
(ii) If $G \in \mathcal{O}_p$ and $e \in E(G)$, then $G/e \in \mathcal{O}_p$.
(iii) If $H \subseteq G$ satisfying $H \in \mathcal{O}_p$ and $G/H \in \mathcal{O}_p$, then $G \in \mathcal{O}_p$.
(iv) Every graph in $\mathcal{O}_p$ contains $(p-1)$ edge-disjoint spanning trees.
(v) $mK_2 \in \mathcal{O}_p$ if and only if $m \geq p-1$.
(vi) Let $G = C_n(i_1, i_2, ..., i_n)$. If for each $j \in \mathbb{Z}_n$, $i_j \leq p-1$, and if $\sum_{j=1}^{n} i_j \geq (n-1)(p-1)$, then $G \in \mathcal{O}_p$.

[2] J.-B. Liu, P. Li, J. Li and H.-J. Lai, On weighted modulo orientation of graphs, submitted

Jaeger proposes the following conjecture, whose truth would imply Tutte's 5-flow Conjecture.

Jaeger's [3]

Every 9-edge-connected multigraph admits a modulo 5-orientation.

[3] F. Jaeger, Nowhere-zero flow problems, in: Selected Topics in Graph Theory, vol. 3, L. Beineke and R. Wilson, eds., Academic Press, London, New York, 1988, pp. 91–95.

[4] M. Han, H.-J Lai and J.-B Liu, Modulo 5-orientations and degree sequences, Discrete Applied Math., 260 (2019), 155–163.

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Theorem ([4], 2019)

For any multigraphic sequence $d = (d_1, d_2, \cdots, d_n)$ with $\min_{i \in [n]} d_i \geq 9$, d has a 9-edge-connected modulo 5-realization.

[3] F. Jaeger, Nowhere-zero flow problems, in: Selected Topics in Graph Theory, vol. 3, L. Beineke and R. Wilson, eds., Academic Press, London, New York, 1988, pp. 91–95.

[4] M. Han, H.-J Lai and J.-B Liu, Modulo 5-orientations and degree sequences, Discrete Applied Math., 260 (2019), 155–163.

Theorem ([2], 2019+)

Let $p > 0$ be an odd prime, and let G be a graph with Euler genus g and edge connectivity

$$\kappa'(G) \geq \begin{cases} 4p - 6 + \lfloor g/2 \rfloor & g \leq 2, \\ (p-2)\lfloor \sqrt{6g + 0.25} + 2.5 \rfloor + 1 & g \geq 3, \\ p\sqrt{4.98g} & g \text{ is sufficiently large.} \end{cases}$$

Then G admits an $(f, b; p)$-orientation.

[2] J.-B. Liu, P. Li, J. Li and H.-J. Lai, On weighted modulo orientation of graphs, submitted

- An additive basis of $\mathbb{Z}_n^p$ is a multiset $\{x_1, x_2, \ldots, x_m\}$ of $\mathbb{Z}_n^p$ such that for any $x \in \mathbb{Z}_n^p$, there exist scalars $c_i \in \{0, 1\}$ such that

$$x = \sum_{i=1}^{m} c_i x_i.$$

- $c(n, p)$ is the smallest positive integer t such that for any t (linear) bases $B_1, \ldots, B_t$ of $\mathbb{Z}_n^p$, the union (with repetitions) $\cup_{i=1}^{t} B_i$ forms an additive basis of $\mathbb{Z}_n^p$.

Theorem

Let p be a prime at least 3.
(i) (Davenport [5], see also [6]) $c(1, p) = p - 1$.
(ii) (Mann and Wou [7]) $c(2, p) = p - 1$.
(iii) (Alon, Linial and Meshulam [8])
$c(n, p) \le (p - 1)\log n + p - 2$.

[5] H. Davenport, On the addition of residue classes, J. London Math. Soc., 10 (1935), 30–32.

[6] N. Alon, M. Nathanson and I. Ruzsa, The polynomial method and restricted sums of congruence classes, J. Number Theory, 56(2) (1996), 404–417.

[7] H. B. Mann and Y. F. Wou, An addition theorem for the elementary abelian group of type (p; p), Monatsh Math., 102 (1986), 273–308.

[8] N. Alon, N. Linial and R. Meshulam, Additive bases of vector spaces over prime fields, J. Combin. Theory Ser. A, 57(1991), 203–210.

Theorem ([9,2019+])

Let p be a prime at least 3.
(i)Let K_n be a complete graph. If $n \geq 2(p-1)(5+3\log(p-1))$, then $K_n \in \mathcal{O}_p$.
(ii) Let G be a connected chordal graph. If $\kappa(G) \geq 2(p-1)(5+3\log(p-1)) - 1$, then $G \in \mathcal{O}_p$.
(iii) Let $n_1 = \frac{1}{2}(p-1)(p-2)+1$ and $n_2 = \frac{1}{2}n_1(n_1-1)(p-1)$. Then $G = K_{n_1,n_2} \in \mathcal{O}_p$.

[8] J.-B. Liu, M. Han, J. Li and H.-J Lai, On weighted modulo orientations of certain graphs, submitted.

A signed graph is an ordered pair (G, σ) consisting of a graph G with a mapping $\sigma : E(G) \to \{1, -1\}$. An edge $e \in E(G)$ is positive if $\sigma(e) = 1$ and negative if $\sigma(e) = -1$.

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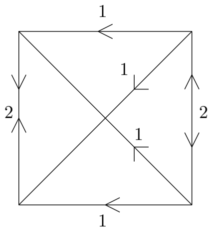

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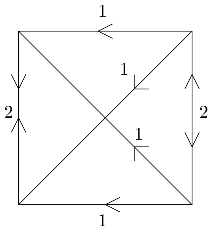

Theorem ([9,2019+])

Let p be an odd prime and let (G, σ) be a $(p-1)$-unbalanced signed graph with $\kappa'(G) \geq 12p^2 - 28p + 15$. Then $(G, \sigma) \in \mathcal{O}_p$.

We believe that edge-connectivity of G is a linear function of p would suffice. We conclude the following conjectures.

Conjecture

There exists a constant c independent of p such that every cp-edge-connected graph has an $(f, b; p)$-orientation.

Thank You!