Dimensions of metric spaces

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Content

Dimensions

There are many concepts of dimensions of a topological space.

V.V. Fedorchuk

The Fundamentals of Dimension Theory, in General Topology I, Springer-Verlag, Berlin-Heidelberg 1990.

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L.M. Blumenthal

Theory and Applications of Distance Geometry. Clarendon Press, Oxford (1953).

Metric dimension

Let X be a metric space with distance function $\rho : X \times X \to [0, \infty)$. Let $A \subseteq X$. If for every $x, y \in X$, $x \neq y$ implies there exists $a \in A$ such that $\rho(a, x) \neq \rho(a, y)$ then A is said to resolve X, and is called a resolving set or briefly a resolver for X. A resolving set of minimum cardinality is called a metric basis for X. The cardinality of a minimum resolving set is called the metric dimension of X and is denoted $\beta(X)$. Note that the condition for A to be resolving may be written in a logically equivalent form:

$$[\forall a \in A, \rho(a, x) = \rho(a, y)] \Rightarrow x = y.$$

This was the definition given by Blumenthal in his monograph of 1953.

Distance between sets

Let X be a metric space with distance function ρ. Let $A, B \subseteq X$. Define the distance between the sets A and B to be

$$\rho(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}.$$

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Partition dimension

Let $\mathscr{A} = \{A_1, A_2, \ldots, A_n, \ldots\}$ be a partition of X, with $A_i \subseteq X$ for every $i = 1, 2, \ldots, n, \ldots$. If

$$x \neq y \Rightarrow \exists i \geq 1, \rho(A_i, x) \neq \rho(A_i, y),$$

then the partition $\mathscr{A}$ is said to resolve X. If $\mathscr{A}$ resolves X and the cardinality $|\mathscr{A}|$ is minimum, then the cardinality $\beta_p = |\mathscr{A}|$ is called the partition dimension of X.

Definition

For each $i = 1, 2, \ldots, n, \ldots$, let $A_i \subseteq X$ and

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- If each subset $A_i = \{a_i\}$ is a singleton, then we have $\delta(X) = \beta(X)$. Hence $\delta(X)$ is a generalization of metric dimension.
- If $\mathscr{A}$ is a partition of X, then $\delta(X) = \beta_p(X)$, and hence $\delta(X)$ generalizes the partition dimension.

Bisectors

Bisector

Let X be a metric space with distance function ρ. Let $u, v \in X$. Define the bisector of u, v to be the set

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Bisector of sets

Let X be a metric space with distance function ρ. Let $U, V \subseteq X$. Define the bisector of U, V to be the set

$$B(U, V) = \{x \in X : \rho(U, x) = \rho(V, x)\}.$$

Since $A \subseteq X$ is not resolving if and only if there exist $u, v \in X$ with $u \neq v$ such that for every $a \in A$, $\rho(a, u) = \rho(a, v)$, $A \subseteq X$ resolves X if and only if no bisector contains A. This proves

Proposition

Let X be a metric space and $A \subseteq X$. Then A with does not resolve X if and only if there exist $u, v \in X$ with $u \neq v$ such that $A \subseteq B(u, v)$.

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$$B(x, y) = \{z : |z - x| = |z - y|\}.$$

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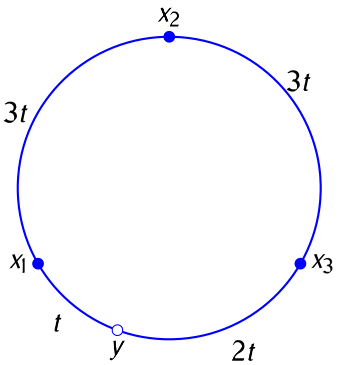
- Under the present context, a conic section can be a bisector: let l be any straight line and a be any fixed point not on l; then by the geometric definition of the parabola, $B(a, l)$ is the parabola which is the locus of all point whose distance to a is equal to its distance to l.

Monotonicity

Monotonicity is a natural axiom for a well defined concept of a dimension: if X is a subspace of Y then it is natural to require that $\dim X \leq \dim Y$. The metric dimension fails to satisfy this natural axiom. We show this with an example. Let $t > 0$.

Figure 1: An isometric subspace with a higher dimension



- Let $t > 0$ and $X = \{x_1, x_2, x_3\}$ with

$$\rho_X(x_1, x_2) = \rho_X(x_1, x_3) = \rho_X(x_2, x_3) = 3t.$$

and $Y = X \cup \{y\}$ with

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$$\rho_Y(x_1, y) = t,\ \rho_Y(x_2, y) = 4t,\ \rho_Y(x_3, y) = 2t.$$

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- Hence X is isometrically embedded in Y (X is an isometric subspace of Y).
- $\{y\}$ is a metric basis for Y.
- No set with one element resolves X and $\{x_1, x_2\}$ resolves X.

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- Hence X is isometrically embedded in Y (X is an isometric subspace of Y).
- $\{y\}$ is a metric basis for Y.
- No set with one element resolves X and $\{x_1, x_2\}$ resolves X.
- Hence $\beta(X) = 2$ and $\beta(Y) = 1$.

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- Hence $\beta(X) = 2$ and $\beta(Y) = 1$.
- We have seen now that the concept of metric dimension is a peculiar concept. This concept is a weird concept.

Finite resolution

Let X be the metric space of $\mathbb{Z} \times \mathbb{Z}$ determined by the finite generating set

$$S = \{u \in \mathbb{Z} \times \mathbb{Z} : |u| = 1\}.$$

- Then $|S| = 4$ and $S = \{(1,0),(0,1),(-1,0),(0,-1)\}$.

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- Then there exists a rectangle R with top right vertex (p, q), of finite integer side lengths such that $A \subseteq R$.

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- Then if $x = (p+1, q)$ and $y = (p, q+1)$, then for each $a \in A$, $\rho(a, x) = \rho(a, y)$. (Every geodesic from a to x and to y contains (p, q).)

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- Hence A does not resolve X.

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- Hence A does not resolve X.
- This shows that X is not finitely resolved.

We obtained

Theorem

If Y is the metric space of a finitely generated torsion-free abelian group and X is an isometric subspace of Y then $\beta(X) \leq \beta(Y)$.

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Which metric spaces and their isometric subspaces satisfy the monotonicity axiom?

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3. **Spherical spaces** $\mathbb{S}^n$ is the set $\{\mathbf{x} \in \mathbb{R}^{n+1} : \|\mathbf{x}\| = 1\}$ with the path metric induced by the Euclidean metric in $\mathbb{R}^{n+1}$.

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The idea of a proof will be illustrated through an example.

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- Let $u, v \in \mathbb{S}^2$ with $u \neq v$.
- Then there exists a unique plane through the three points $(0, 0, 0)$, u and v (Euclid stated this as an axiom).

- This plane intersects X in a circle with the same radius 1.

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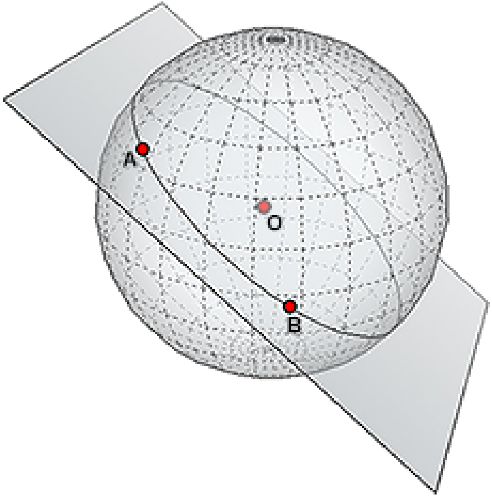
- The bisector $B(u, v)$ passing through the midpoint of geodesic arc $\gamma(u, v)$ connecting u and v, which is perpendicular to the geodsic arc $\gamma(u, v)$, is a great circle (a geodesic line).



Great circle and geodesic arc.

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- Every bisector in X is a great circle.
- Since $\{a, b, c\}$ is not contained in a same great circle, by Proposition on resolution, $\{a, b, c\}$ resolves X. $\square$

Riemann surfaces

Another main result of the paper by Bau and Beardon is

Theorem

Every Riemann surface S with its path metric satisfies $\beta(S) = 3$.

We shall not sketch a proof of this result but we pass on by mentioning that our proof used the uniformization theorem. This was proved towards the middle of the twentieth century and is now a well-known basic theorem in complex function theory.

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Uniformization Theorem

If S is a Riemann surface then S is the quotient of a metric space X by the action of a discrete group G of isometries of X, where X is the Euclidean, hyperbolic, or the Riemann sphere. If X is the Euclidean (or complex) plane $\mathbb{C}$, then S is $\mathbb{C}$, a cylinder or a torus. If X is the Riemann sphere, then S is X. If X is the hyperbolic plane, then S is the quotient of the hyperbolic plane by some discrete group action.

Metric spaces of semigroups

Cayley graphs of groups

Let G be a group and $\mathcal{S} \subseteq G$ be a generating set for G such that $1 \notin \mathcal{S}$, $\mathcal{S}^{-1} = \mathcal{S}$. Define the Cayley graph $X = X(G, \mathcal{S})$ by the specification

$$V(X) = G, \; E(X) = \{gh : g, h \in G, gh^{-1} \in \mathcal{S}\}.$$

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Torsion

Let G be a group. If for $x \in G$ there exists $n \in \mathbb{N}$ such that $x^n = 1$ then by the well ordering principle, there exists a smallest positive integer n such that $x^n = 1$. The smallest positive integer n for which $x^n = 1$ is called the order of x and x is called a torsion element (element of finite order). Note that the identity element is always a torsion element of order 1.

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Primitive torsionfreie abelsche Gruppen von endlichen Range, Math. Ann., **38**(1937), 175-203.

If an additive abelian group is under consideration, we use additive notation. The identity element for addition is called the zero element and is denoted 0. The binary operation is denoted $+$, and the inverse of an element x is its negative and is denoted $-x$. The condition imposed on the generating set $\mathcal{S}$ now becomes $0 \notin \mathcal{S}$ and $-\mathcal{S} = \mathcal{S}$. The torsion condition in additive notation is: there exists $n \in \mathbb{N}$ such that $nx = \underbrace{x + x + \cdots + x}_{n} = 0$.

Example

Let $X = (\mathbb{Z} \times \mathbb{Z}, \mathcal{S})$ with $\mathcal{S} = \{x \in \mathbb{Z} \times \mathbb{Z} : |x| = 1\}$. We consider a few examples of bisectors.

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- Let $k, l \in \mathbb{Z}$, and let
$$Y = \{(k, y) : y \in \mathbb{Z}\}, \ Z = \{(l, z) : z \in \mathbb{Z}\}.$$
- If $k \equiv l \pmod 2$, then we have that
$$B(Y, Z) = \left\{ \left(\frac{k+l}{2}, w \right) : w \in \mathbb{Z} \right\}.$$

- If $k \not\equiv l \pmod 2$, then $B(Y, Z) = \emptyset$.

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- Consider $A = \{(0, x) : x \in \mathbb{Z}\}$ and $a = (k, 0)$.
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$$B(a, A) = \{(x, \pm(2x - k)) : 0 \leq x < k\} \cup \{(x, \pm k) : x \geq k\}.$$

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- With $k = 3$, we illustrate this in the figure below.

- If $k \not\equiv l \pmod 2$, then $B(Y, Z) = \emptyset$.
- Consider $A = \{(0, x) : x \in \mathbb{Z}\}$ and $a = (k, 0)$.
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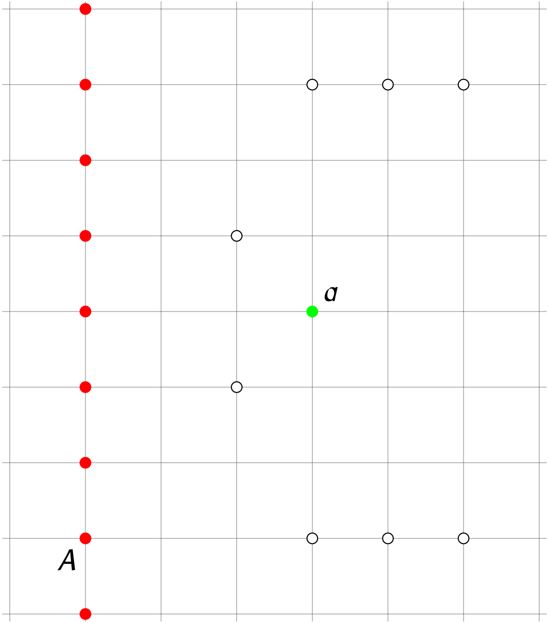
- Note that this is the parabola in X according to its geometric definition and according to the metric of X.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- Let

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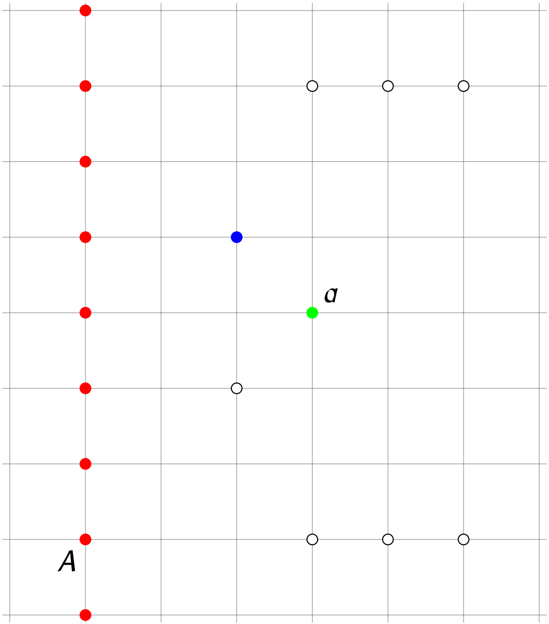
- It is also straightforward to verify that $\{a,A,B\}$ also resolves X.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