When Cayley graphs are wreath products.

6th Annual Mississippi Discrete Mathematics Workshop

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Definition
Let G be a group and $S \subset G$ such that $1 \notin S$ and $S = S^{-1}$. Define a **Cayley digraph of G**, denoted $\mathrm{Cay}(G, S)$, to be the graph with vertex set

$$V(\mathrm{Cay}(G, S)) = G$$

and edge set

$$E(\mathrm{Cay}(G, S)) = \{(g, gs) : g \in G, s \in S\}.$$

We call S the **connection set of $\mathrm{Cay}(G, S)$**.

Cayley Graphs

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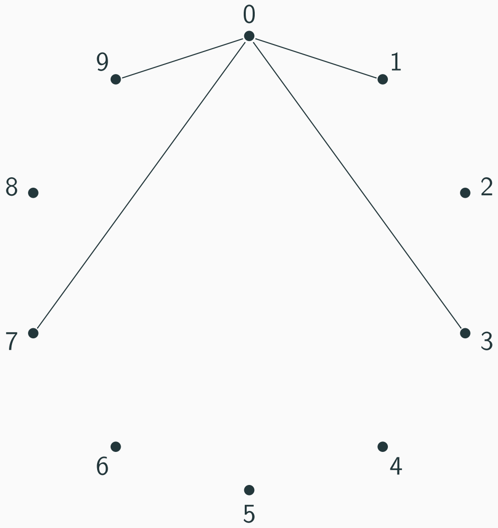
So, the vertices of a Cayley graph are the elements of the group (and we label the vertices using the group elements), and the edge set is determined by the connection set S.

The Cayley graph $\mathrm{Cay}(\mathbb{Z}_{10}, \{1, 3, 7, 9\})$



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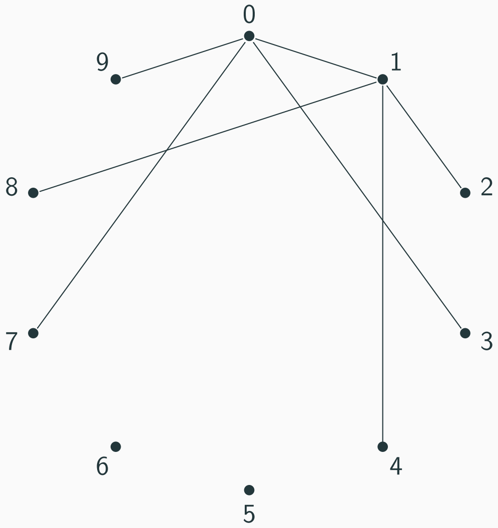

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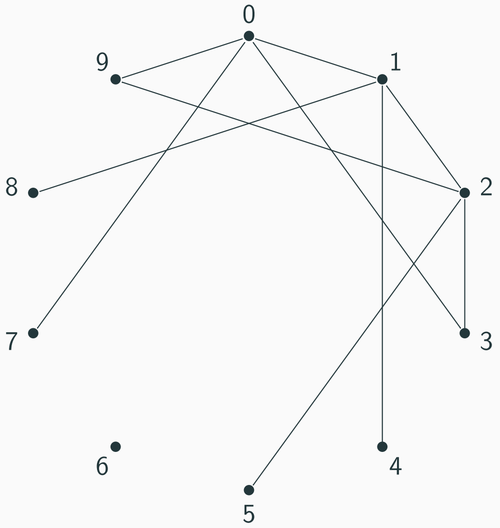

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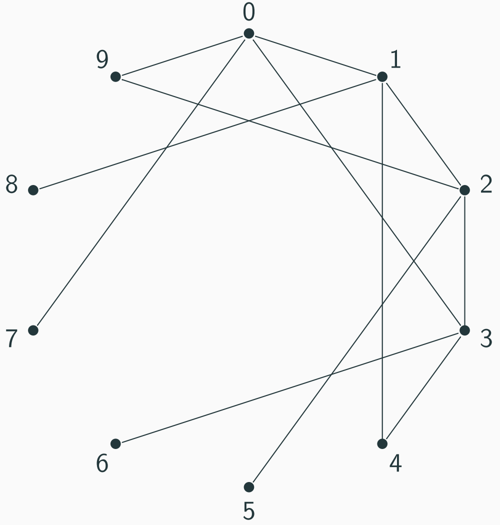

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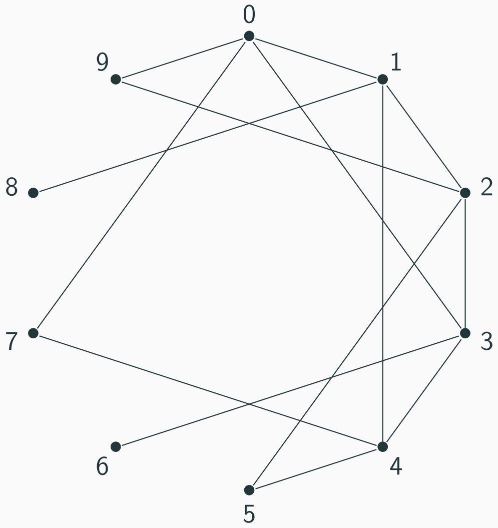

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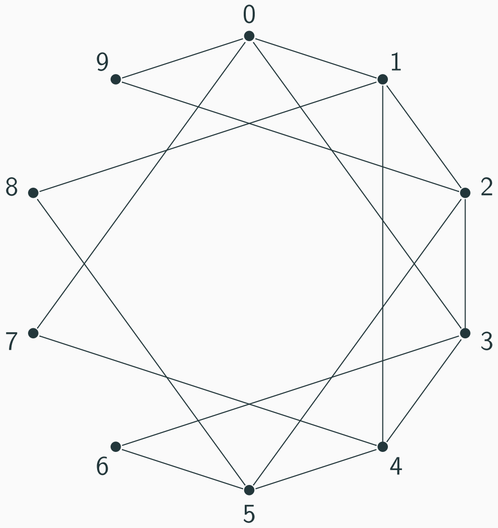

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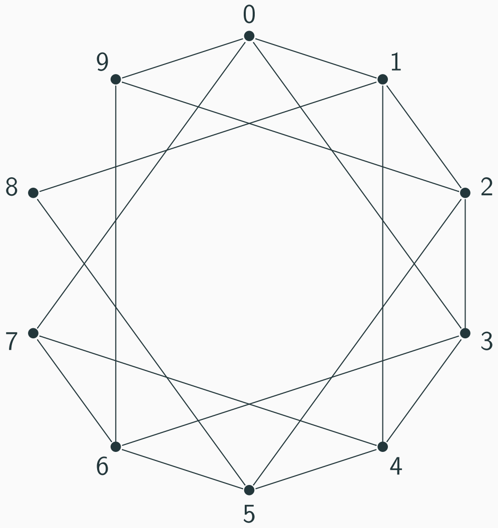

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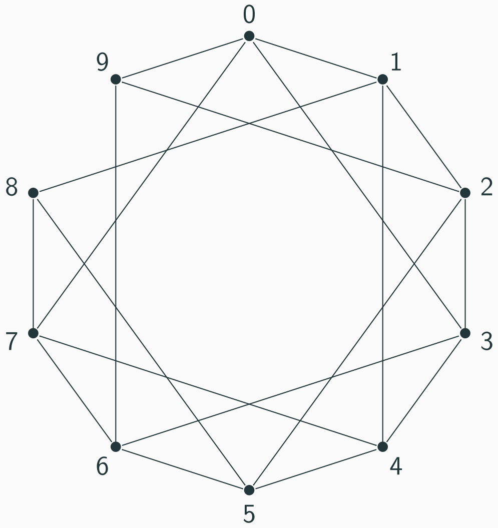

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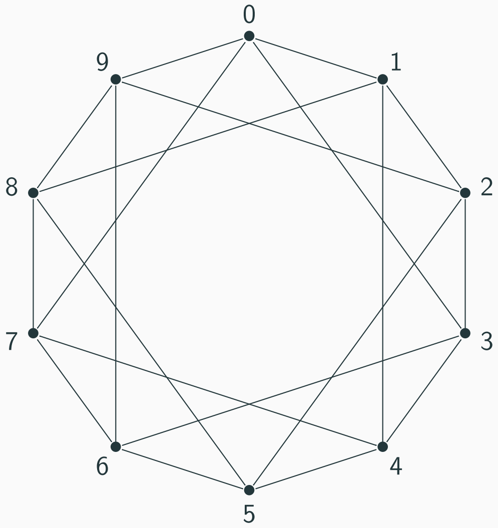

Coset Digraph

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Definition
Let G be a group, $H \leq G$ and $S \subseteq G$. Define a **coset digraph**, denoted $\mathrm{Cos}(G, H, S)$ with vertex set

$$V(\mathrm{Cos}(G, H, S)) = \{gH : g \in G\}, \text{ the set of left cosets of } H \text{ in } G,$$

and arc set

$$A(\mathrm{Cos}(G, H, S)) = \{(xH, yH) :\leftrightarrow x^{-1}y \in HSH\}.$$

The digraph $\mathrm{Cos}(G, H, S)$ is called a **Sabidussi coset digraph of G**.

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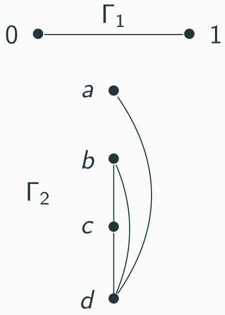

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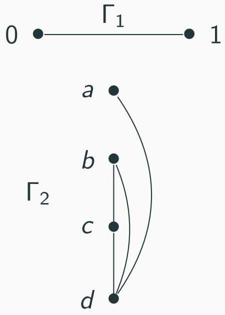

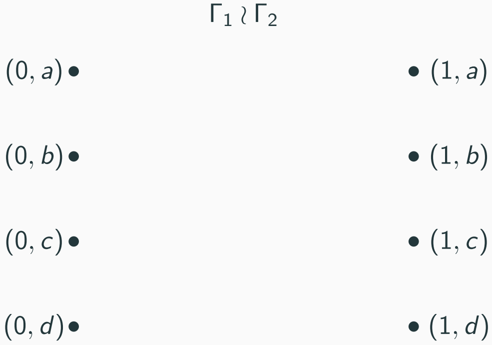

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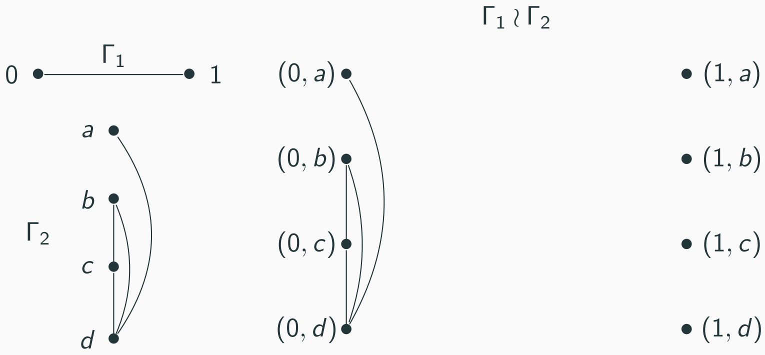

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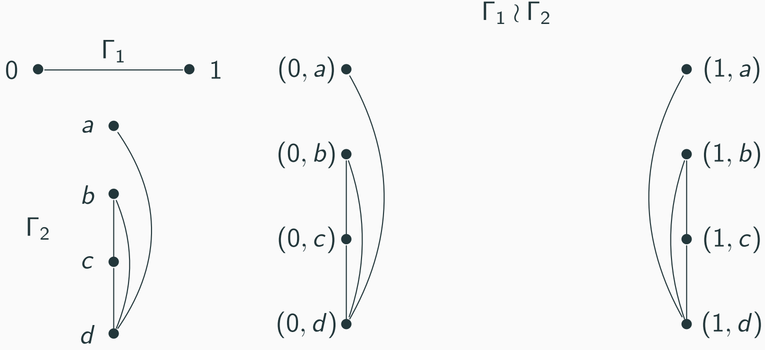

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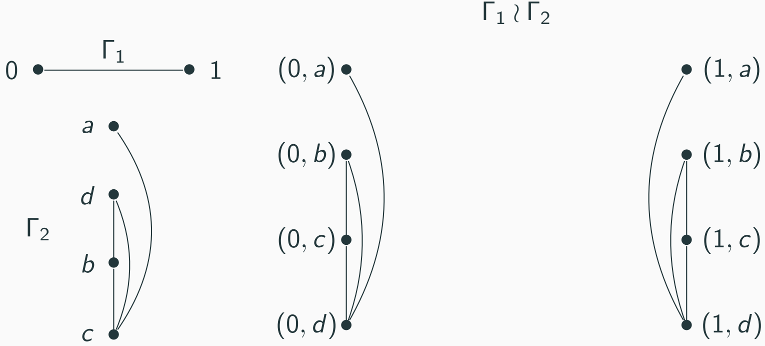

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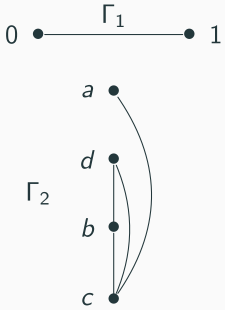

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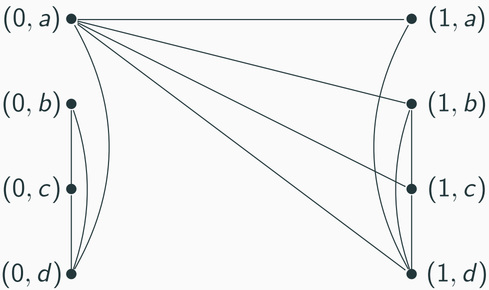

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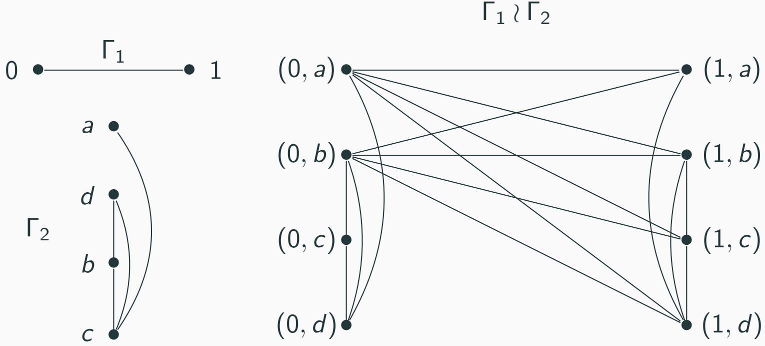

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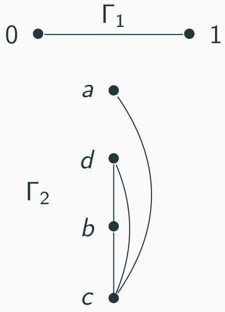

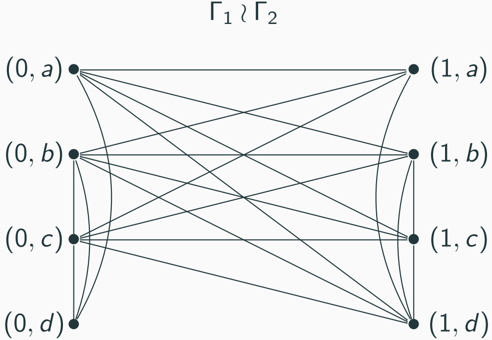

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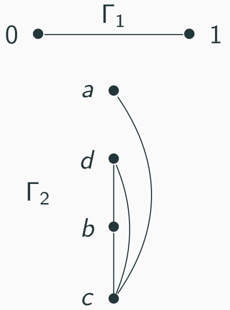

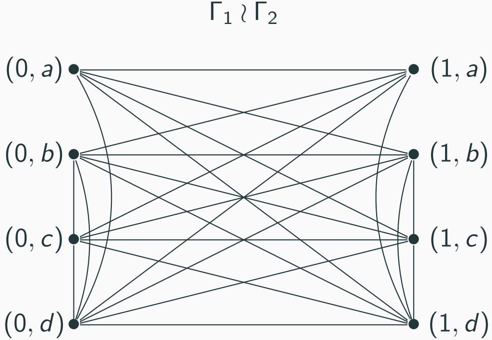

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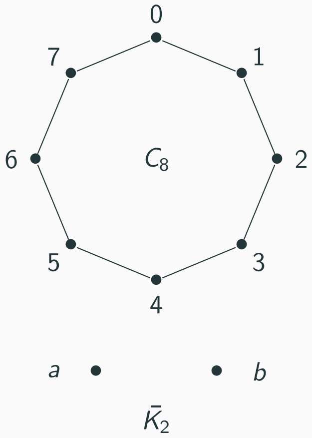

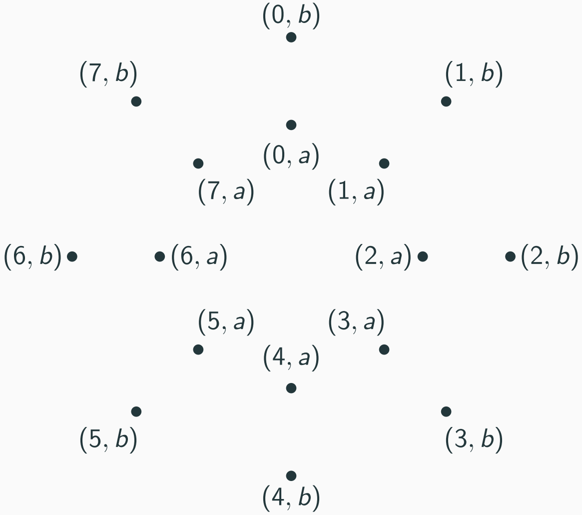
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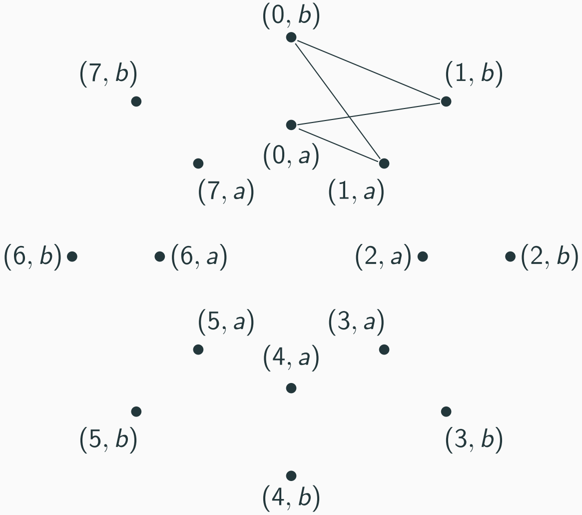

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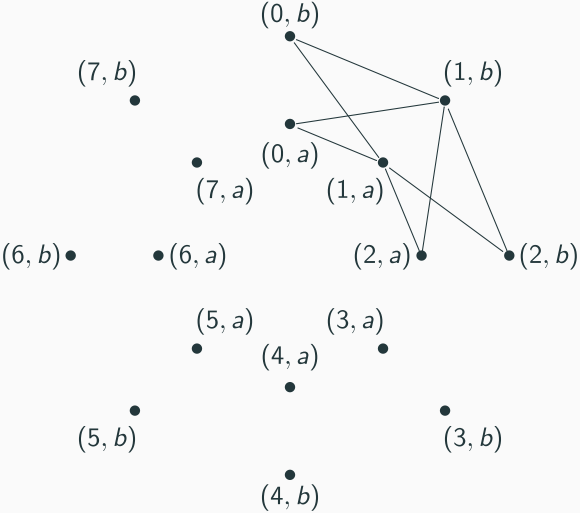

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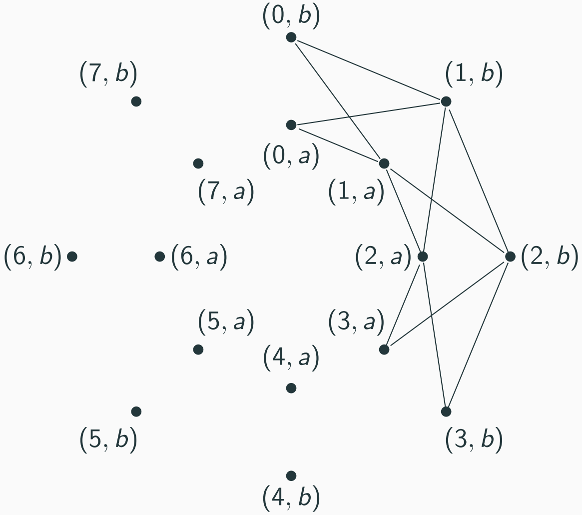

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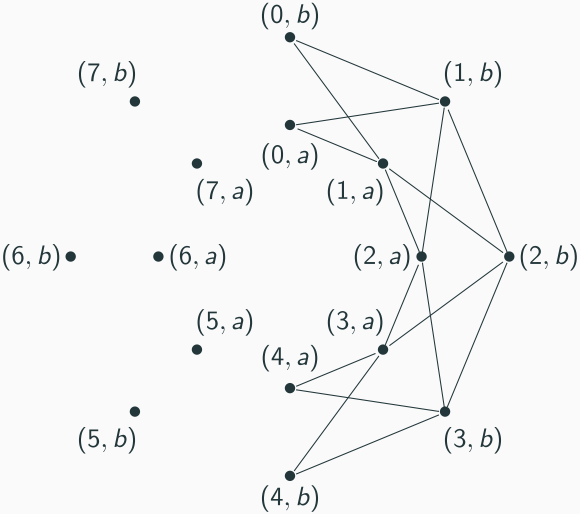

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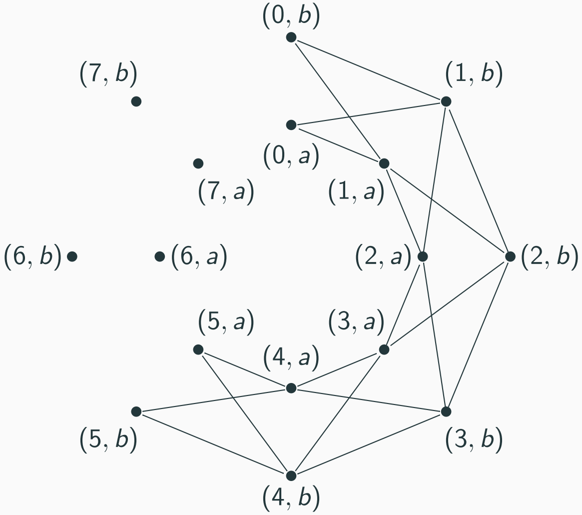

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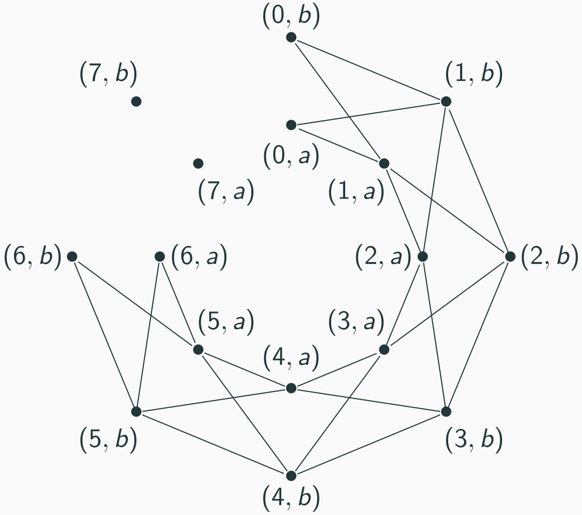

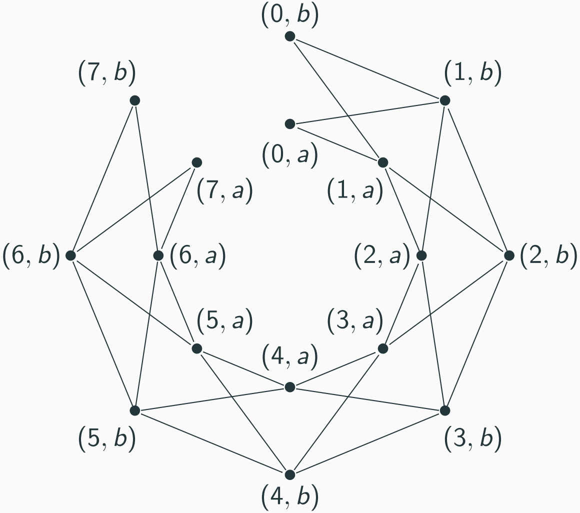
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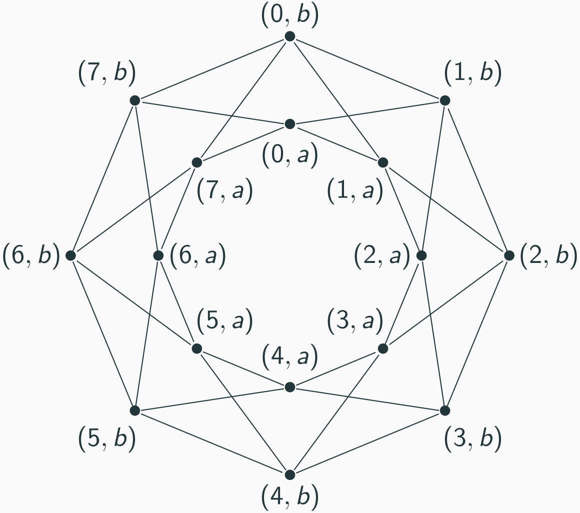
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Cayley Graph with $H \triangleleft G$.

Theorem
A Cayley digraph $\Gamma = \mathrm{Cay}(G, S)$ of a group G is isomorphic to a nontrivial wreath product of two vertex-transitive digraphs of smaller order if there exists $H \triangleleft G$ such that $S - H$ is a union of cosets of H in G. Then,

$$\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G/H, S_1) \wr \mathrm{Cay}(H, S_2)$$

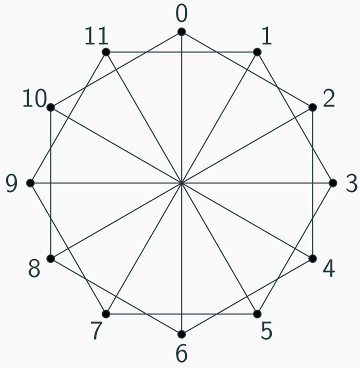
where S_1 is the set of cosets of H contained in S and $S_2 = S \cap H$.

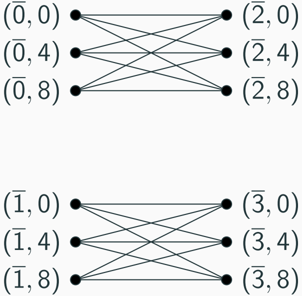
Abelian group example

Example

Let $G = \mathbb{Z}_{12}$ and $H = \langle 4 \rangle \cong \mathbb{Z}_3$. Since G is abelian, $H \triangleleft G$. The graph $\mathrm{Cay}(\mathbb{Z}_{12}, \{2, 6, 10\})$ has connection set the coset $\overline{2} = 2 + \langle 4 \rangle$. Then $\mathrm{Cay}(\mathbb{Z}_{12}, \{2, 6, 10\})$ is isomorphic to the wreath product $\mathrm{Cay}(\mathbb{Z}_{12}/\langle 4 \rangle, \{\overline{2}\}) \wr \mathrm{Cay}(\langle 4 \rangle, \emptyset)$, where $\mathrm{Cay}(\mathbb{Z}_{12}/\langle 4 \rangle, \{\overline{2}\}) \cong \mathrm{Cay}(\mathbb{Z}_4, \{2\}) \cong 2K_2$ and $\mathrm{Cay}(\langle 4 \rangle, \emptyset) \cong \overline{K_3}$. The vertices of the graphs can be identified via the map $(\overline{a}, b) \mapsto a + b \pmod{12}$, where $\overline{a} = a + \langle 4 \rangle$.

$\mathrm{Cay}(\mathbb{Z}_{12}/\{0,4,8\},\{2,6,10\}) \wr \mathrm{Cay}(\{0,4,8\},\emptyset)$





Remark

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This theorem is only really helpful for determining when the Cayley graph of an abelian group is a wreath product as any subgroup is normal. So, we had to consider a more general case, relaxing the condition of the subgroup being normal to being any general subgroup.

Cayley Graph with $H < G$

Theorem

A Cayley digraph $\Gamma = Cay(G, S)$ of a group G is isomorphic to a nontrivial wreath product of two vertex-transitive digraphs of smaller order if and only if there exists $H < G$ such that $S - H$ is a union of double cosets of H in G. If such an $H < G$ exists, then

$$Cay(G, S) \cong Cos(G/L, H/L, T) \wr Cay(H, S \cap H),$$

where L is the subgroup of G which fixes left coset of H in G set-wise, and $T = \{(sL)(H/L) : s \in S - H\}$.

Nonabelian group example

Example

Let $G = \mathbb{Z}_2 \times D_3$, where $D_3 = \{\tau, \rho : \tau^2 = \rho^3 = 1; \tau\rho = \rho^2\tau\}$ is the dihedral group with 6 elements. Let $H = \mathbb{Z}_2 \times \langle \tau \rangle$, which is not normal in G as $(0, \rho)(0, \tau)(0, \rho^2) = (0, \rho) \notin \mathbb{Z}_2 \times \langle \tau \rangle$. Consider the Cayley graph

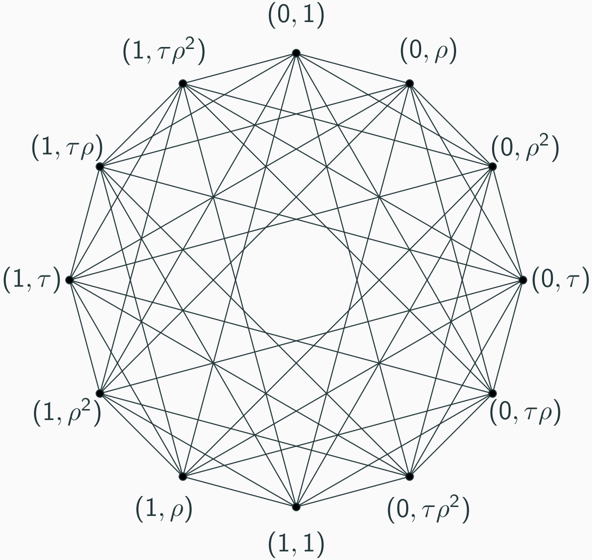
$$\mathrm{Cay}(\mathbb{Z}_2 \times D_3, \{(0, \rho), (1, \rho), (0, \rho^2), (1, \rho^2), (0, \tau\rho), (1, \tau\rho), (0, \tau\rho^2), (1, \tau\rho^2), \}).$$

Denote it by Γ. The connection set of Γ is exactly the double coset $H(0, \rho)H$ and $L = \{(0, 1_{D_3}), (1, 1_{D_3})\}$. So using the theorem, we see that $\Gamma \cong \mathrm{Cos}(G/L, H/L, T) \wr \mathrm{Cay}(H, S \cap H)$, where the set $T = \{(0, \rho)L, (0, \rho^2)L, (0, \tau\rho)L, (0, \tau\rho^2)L\} \subset G/L$. Note that $G/L \cong D_3$, $H/L \cong \langle \tau \rangle$, and $S \cap H = \emptyset$. So

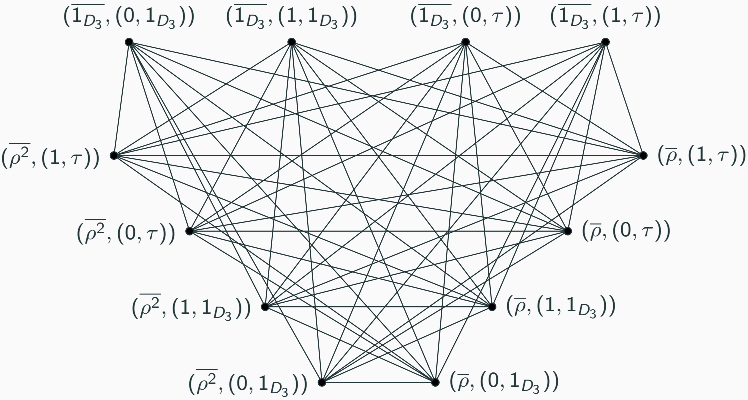
$$\Gamma \cong \mathrm{Cos}(D_3, \langle \tau \rangle, \{\rho, \rho^2, \tau\rho, \tau\rho^2\}) \wr \mathrm{Cay}(\mathbb{Z}_2 \times \langle \tau \rangle, \emptyset) \cong K_3 \wr \overline{K_4}.$$

The graphs can be identified via the map $(\overline{a}, (c, d)) \mapsto (c, ad)$, where $\overline{a}$ is the left coset of H containing a.

Cayley graph of $\mathbb{Z}_2 \times D_3$



$\mathbf{Cos}(D_3, \langle\tau\rangle, \{\rho, \rho^2, \tau\rho, \tau\rho^2\}) \wr \mathrm{Cay}(\mathbb{Z}_2 \times \langle\tau\rangle, \emptyset)$



Research/Next Steps

Question: When are coset digraphs wreath products?

References

1. *Symmetry in Graphs*. Ted Dobson. A book not yet published.