4th Annual Mississippi Discrete Mathematics Workshop

Dimension for Posets and Topological Graph Theory

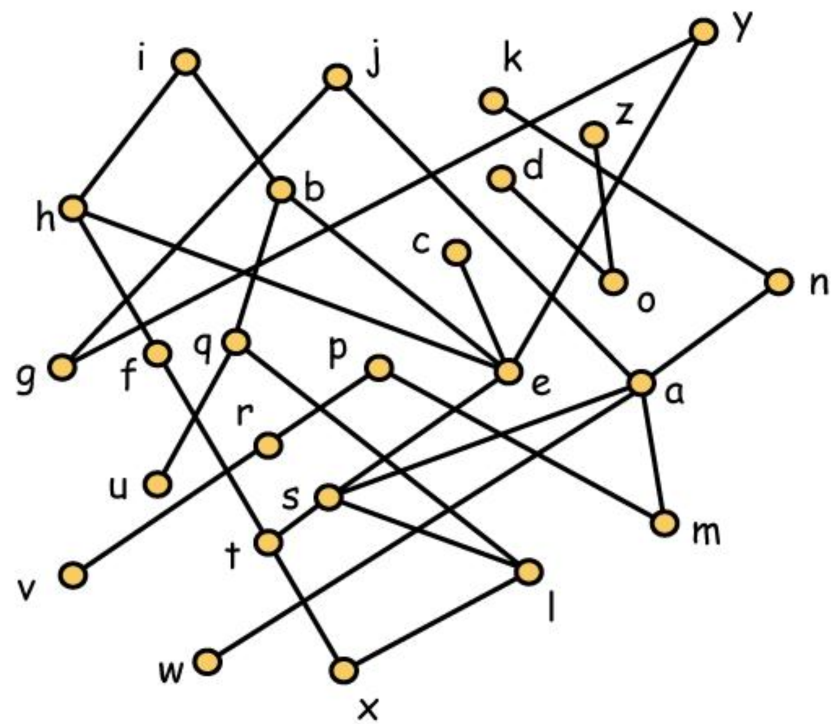
William T. Trotter
trotter@math.gatech.edu

November 14, 2015

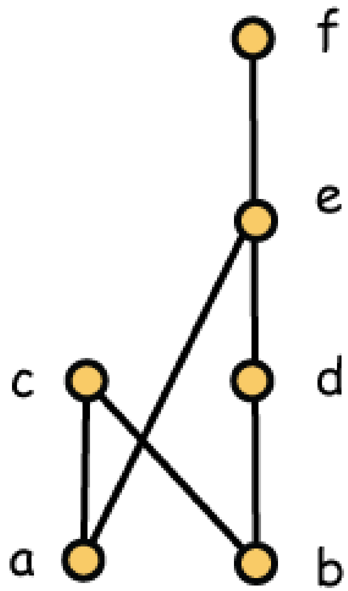
Order Diagram for a Poset on 26 points

Terminology:

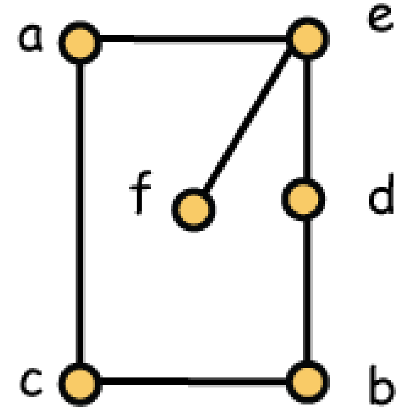
- b < i and s < y.
- j covers a.
- b > e and k > w.
- s and y are comparable.
- j and p are incomparable.
- c is a maximal element.
- u is a minimal element.



Order Diagrams and Cover Graphs

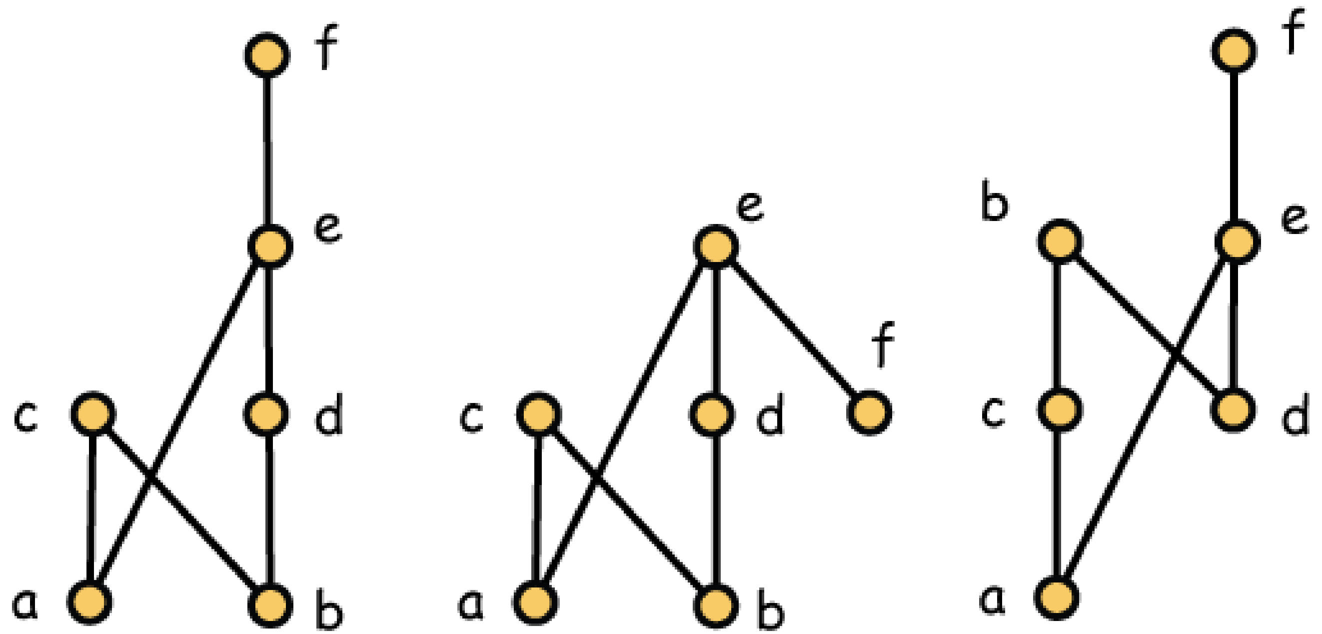


Order Diagram



Cover Graph

Diagrams and Cover Graphs (2)



Three different posets with the same cover graph.

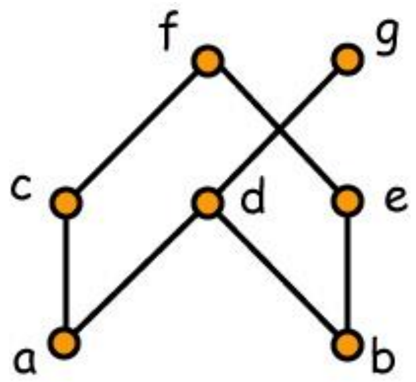
Conventional Wisdom – Until Recently

Observation In general, there are many posets with the same cover graph, and the only poset parameters shared by them are trivial, such as number of elements and number of covering pairs. Other parameters like height, width, number of linear extensions, etc., can differ dramatically.

So the cover graph of a poset doesn't really tell us much about its combinatorial properties.

Realizers of Posets

Definition A family $\mathbf{F} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P is a realizer of P if $P = \cap \mathbf{F}$, i.e., whenever x is incomparable to y in P, there is some L_i in $\mathbf{F}$ with $x > y$ in L_i.



$L_1 = b < e < a < d < g < c < f$

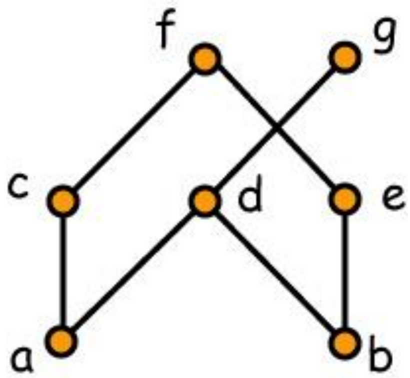
$L_2 = a < c < b < d < g < e < f$

$L_3 = a < c < b < e < f < d < g$

$L_4 = b < e < a < c < f < d < g$

$L_5 = a < b < d < g < e < c < f$

The Dimension of a Poset



$L_1 = b < e < a < d < g < c < f$

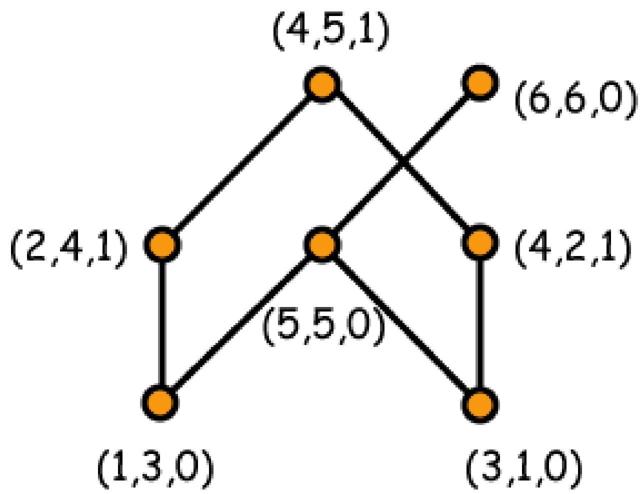
$L_2 = a < c < b < d < g < e < f$

$L_3 = a < c < b < e < f < d < g$

Definition The dimension of a poset is the minimum size of a realizer. This realizer shows $\dim(P) \leq 3$. In fact,

$$\dim(P) = 3$$

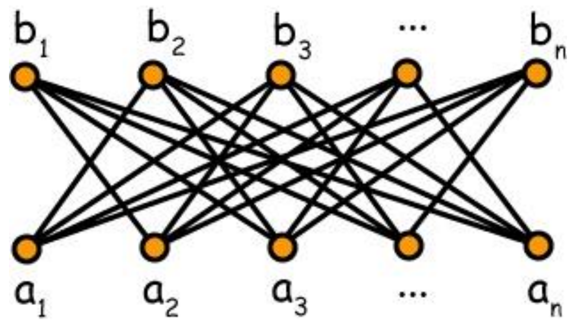
Alternate Definition of Dimension



Remark The dimension of a poset P is the least integer n for which P is a subposet of $\mathbf{R}^n$. This embedding shows that dim(P) ≤ 3. In fact,

$$\dim(P) = 3$$

Standard Examples



S_n

Fact For $n \geq 2$, the standard example S_n is a poset of dimension n. To see that $\dim(S_n) \geq n$, note that if L is a linear extension of S_n, there can only be one value of i for which $a_i > b_i$ in L. To see that $\dim(S_n) \leq n$, use the embedding $a_i = (0,0, \ldots 0, n, 0,0 \ldots,0)$ and $b_i = (n,n, \ldots,n, 0, n, n, \ldots,n)$.

Dimension and Standard Examples

Remark A poset which contains a large standard example has large dimension.

Theorem (Bogart, Rabinovitch and WTT, '76) There are posets with large dimension, not containing the standard example S_2. Such posets must have large height.

Theorem (Felsner and WTT, '00) For every pair (g, d), there is a height 2 poset P such that the girth of the comparability graph of P is at least g and the dimension of P is at least d. Such posets contain S_2 but not S_n when $n \geq 3$.

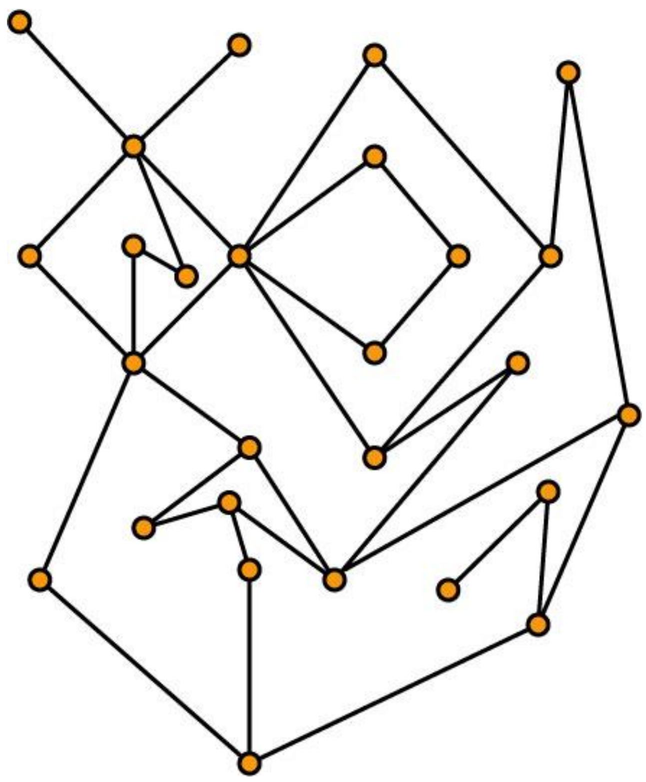
Permutations

Remark A permutation σ on $\{1, 2, \ldots, n\}$ is just a 2-dimensional poset P, where we set $i < j$ in P if and only if $i < j$ in σ and $i < j$ in $\mathbb{N}$.

Remark The incomparability graphs of 2-dimensional posets are just the class of permutation graphs.

Remark The combinatorics of posets really begins once $\dim(P) \geq 3$.

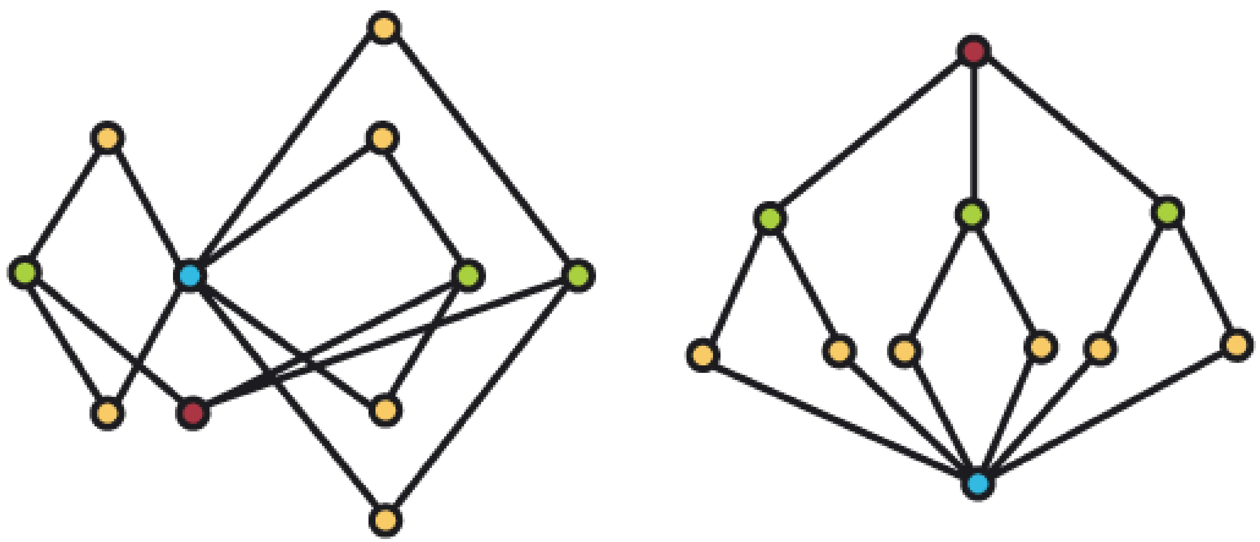
Planar Posets



Definition A poset P is planar when it has an order diagram with no edge crossings.

Fact If P is planar, then it has an order diagram with straight line edges and no crossings.

A Non-planar Poset

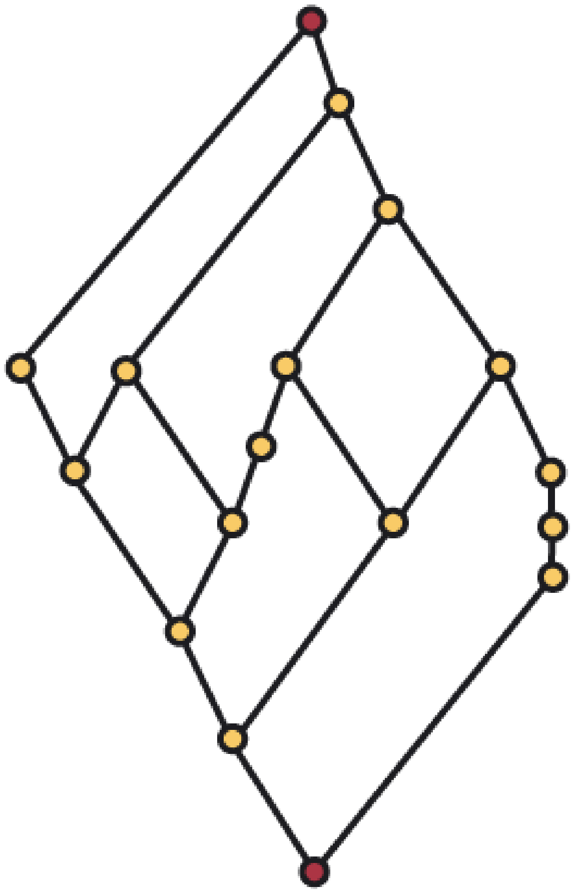


This height 3 non-planar poset has a planar cover graph.

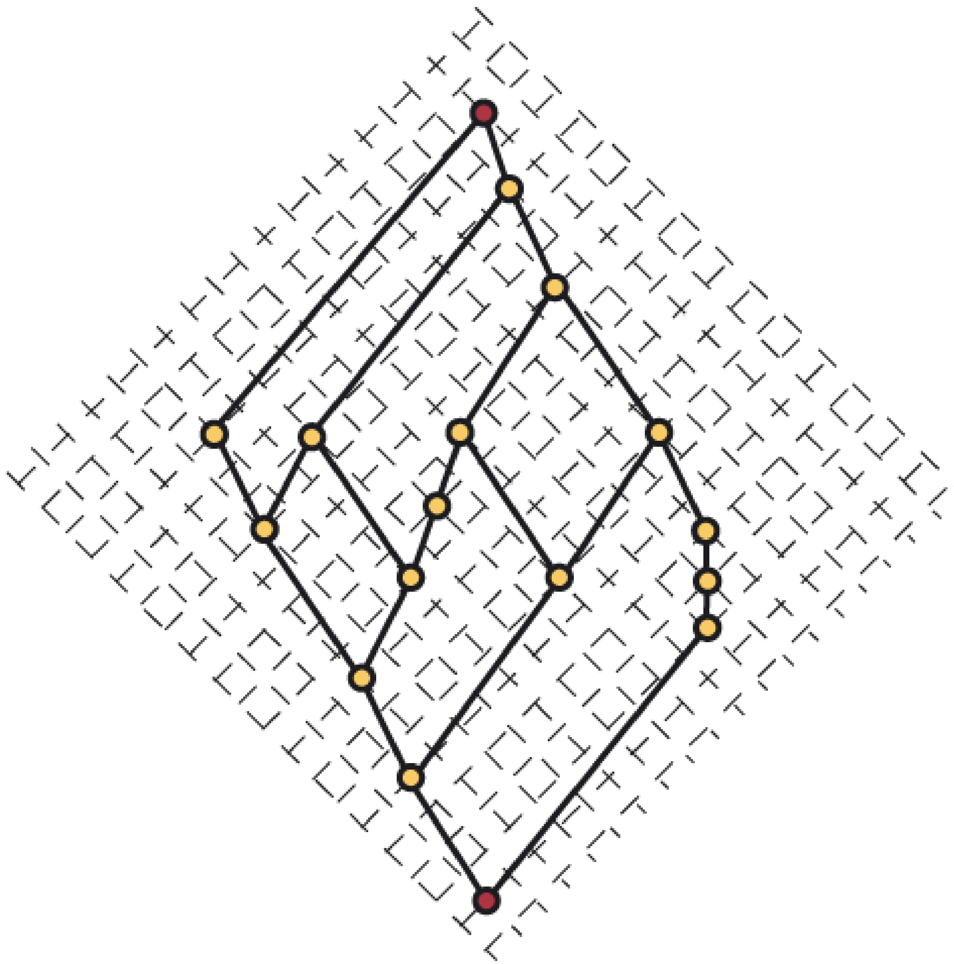
Planar Posets with Zero and One

Theorem (Baker, Fishburn and Roberts, '71 + Folklore)

If P has both a 0 and a 1, then P is planar if and only if it is a lattice and has dimension at most 2.

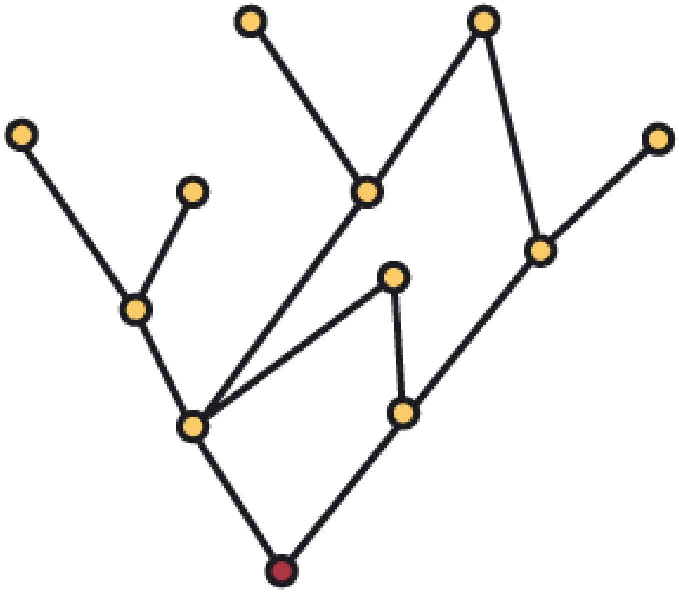


Explicit Embedding on the Integer Grid



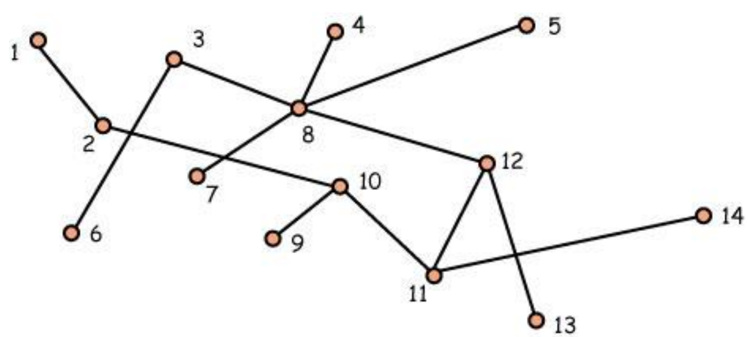
Dimension of Planar Poset with a Zero

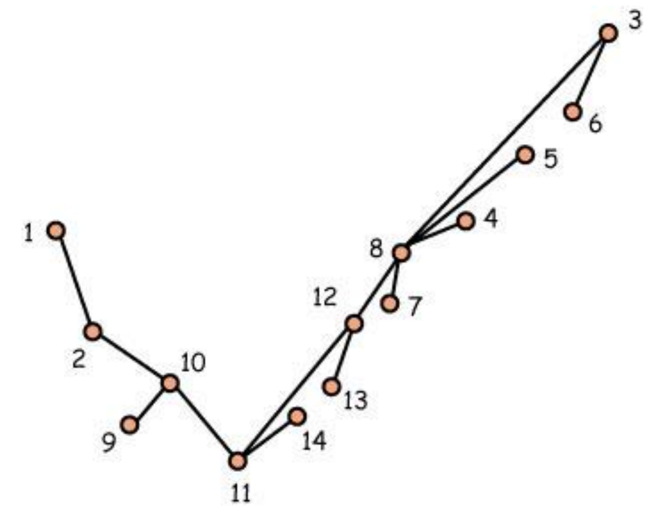
Theorem (WTT and Moore, '77) If P has a 0 and the diagram of P is planar, then $\dim(P) \leq 3$.



The Dimension of a Tree

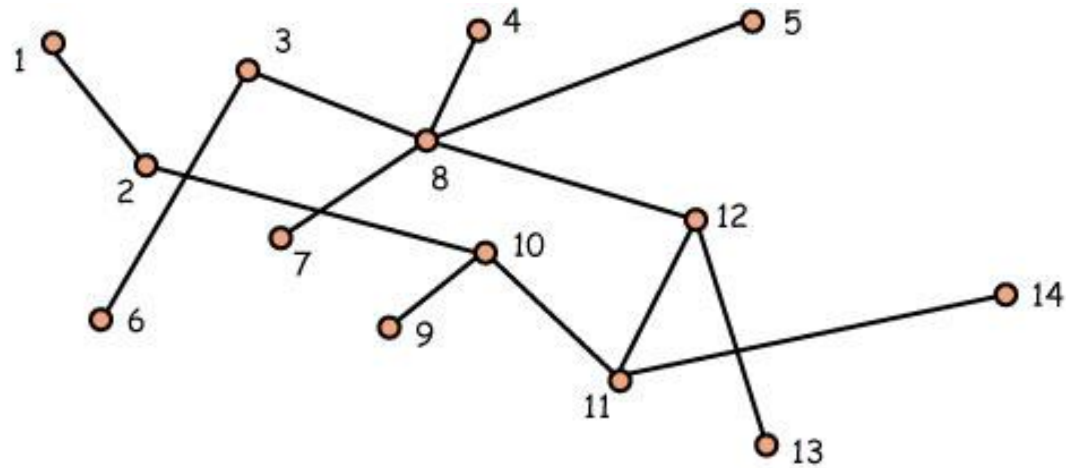
Corollary (WTT and Moore, '77) If the cover graph of P is a tree, then $\dim(P) \leq 3$.





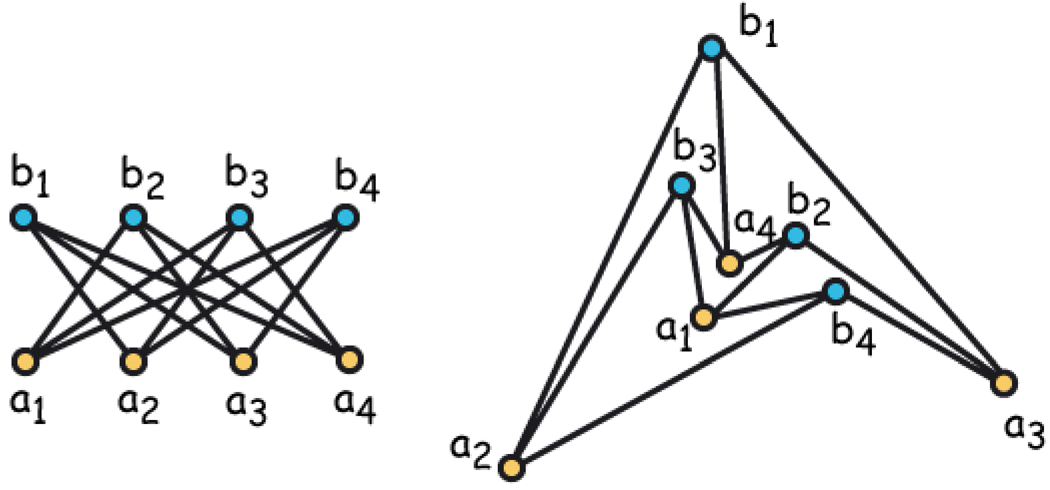
A Restatement - With Hindsight

Corollary (WTT and Moore, '77) If the cover graph of P has tree-width 1, then dim(P) ≤ 3.



A 4-dimensional planar poset

Fact The standard example S_4 is planar!



Fact When $n \geq 5$, the standard example S_n is non-planar.

Wishful Thinking: If Frogs Had Wings …

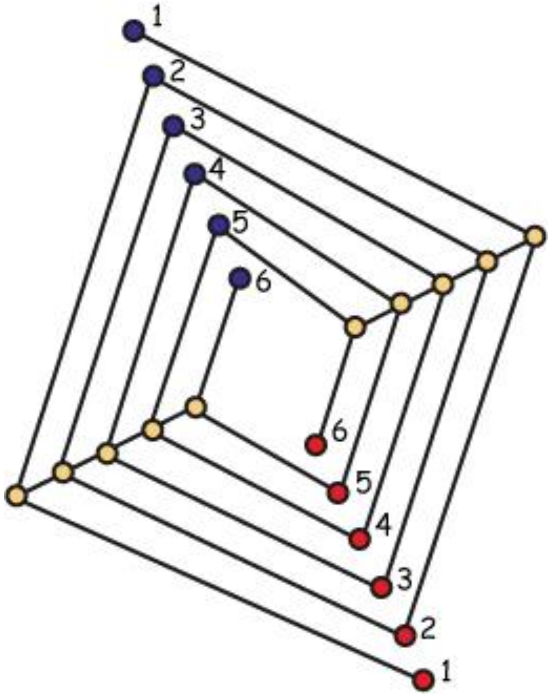
Question Could it possibly be true that dim(P) $\leq$ 4 for every planar poset P?

We observe that

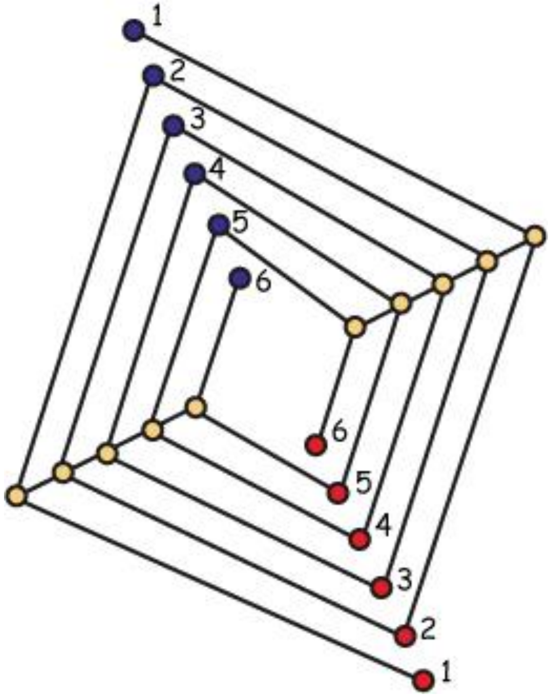
- dim(P) $\leq$ 2 when P has a zero and a one.
- dim(P) $\leq$ 3 when P has a zero or a one.
- So why not dim(P) $\leq$ 4 in the general case?

No … by Kelly's Construction

Theorem (Kelly, '81) For every $n \geq 5$, the standard example S_n is non-planar but it is a subposet of a planar poset.



We Should Have Asked … But Didn't

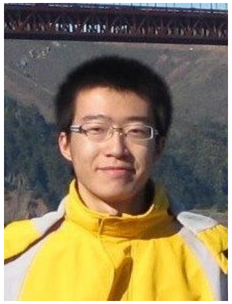


Questions If P is planar and has large dimension, must P contain:

1. Many minimal elements?
2. A long chain?
3. A large standard example?

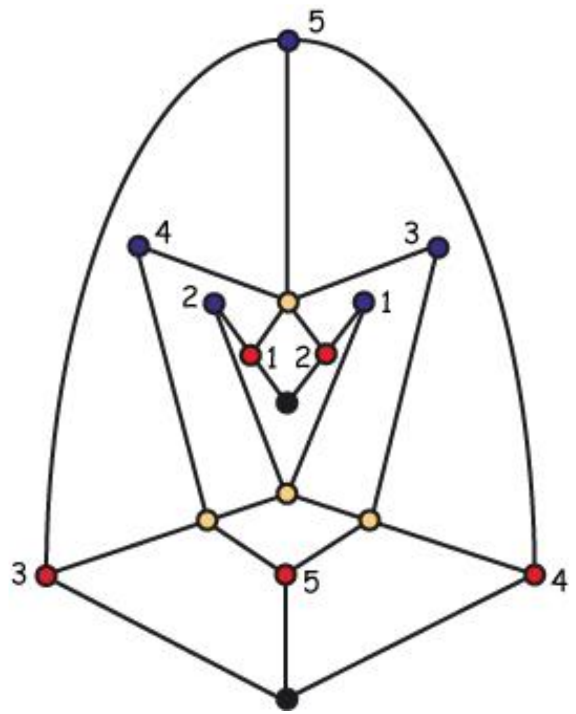
Planar Posets and Minimal Elements

Remark The first of these three questions was posed by R. Stanley in 2013. The answer is "yes" as we were able to prove the following inequality.



Theorem (WTT and Wang, '15) The maximum dimension $m(t)$ of a planar poset with t minimal elements is at most $2t + 1$.

Dimension 5 with 2 Minimal Elements



Remark When $t \geq 3$, we have only been able to show that $m(t) \geq t + 3$.

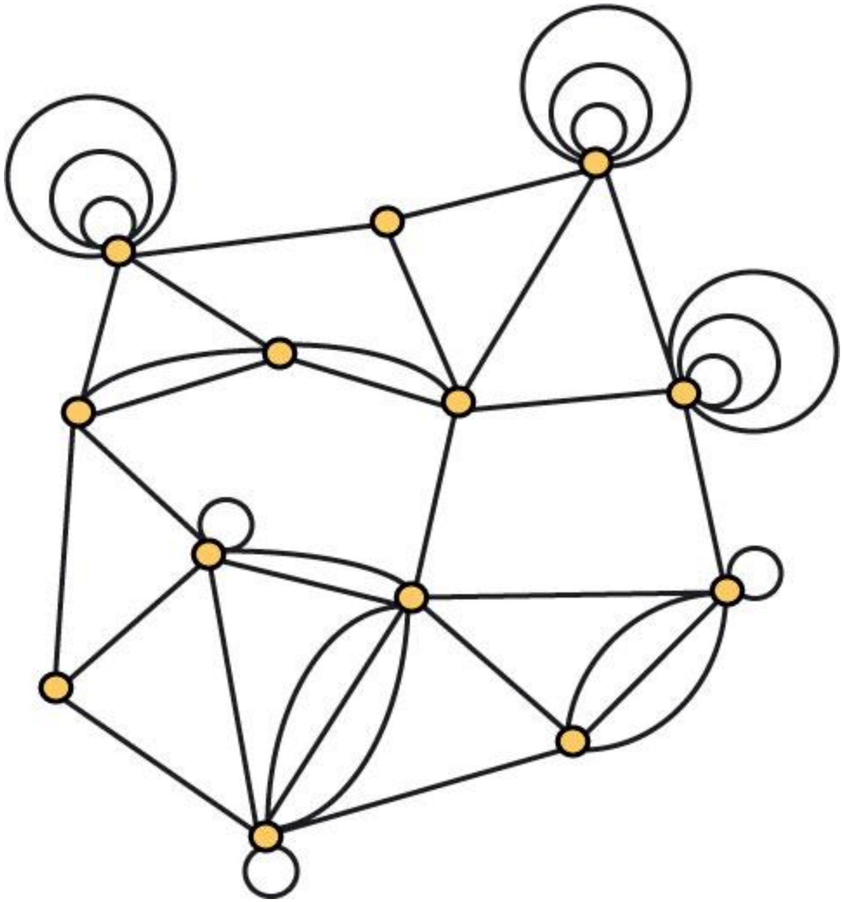
Planar Posets and Planar Cover Graphs

Remark The first question concerns planar posets and **not** posets with planar cover graphs, since the following result was proved some three years before Kelly's construction.

Theorem (WTT, '78) For every $d \geq 1$, there is a poset P with a zero and a one so that $\dim(P) = d$ while the cover graph of P is planar.

Remark But for the rest of the talk, we will be discussing properties of a poset determined in terms of their cover graphs and not their order diagrams, flying in the face of "conventional wisdom."

Planar Multigraphs



Planar Multigraphs and Dimension

Theorem (Brightwell and WTT, '96, '93) Let D be a non-crossing drawing of a planar multigraph G, and let P be the vertex-edge-face poset determined by D. Then $\dim(P) \leq 4$. Furthermore, if G is a simple 3-connected graph, then the subposet of P determined by the vertices and faces is 4-irreducible.

Remark The second statement is stronger than Schnyder's celebrated theorem: *A graph G is planar if and only if the dimension of its vertex-edge poset is at most 3.*

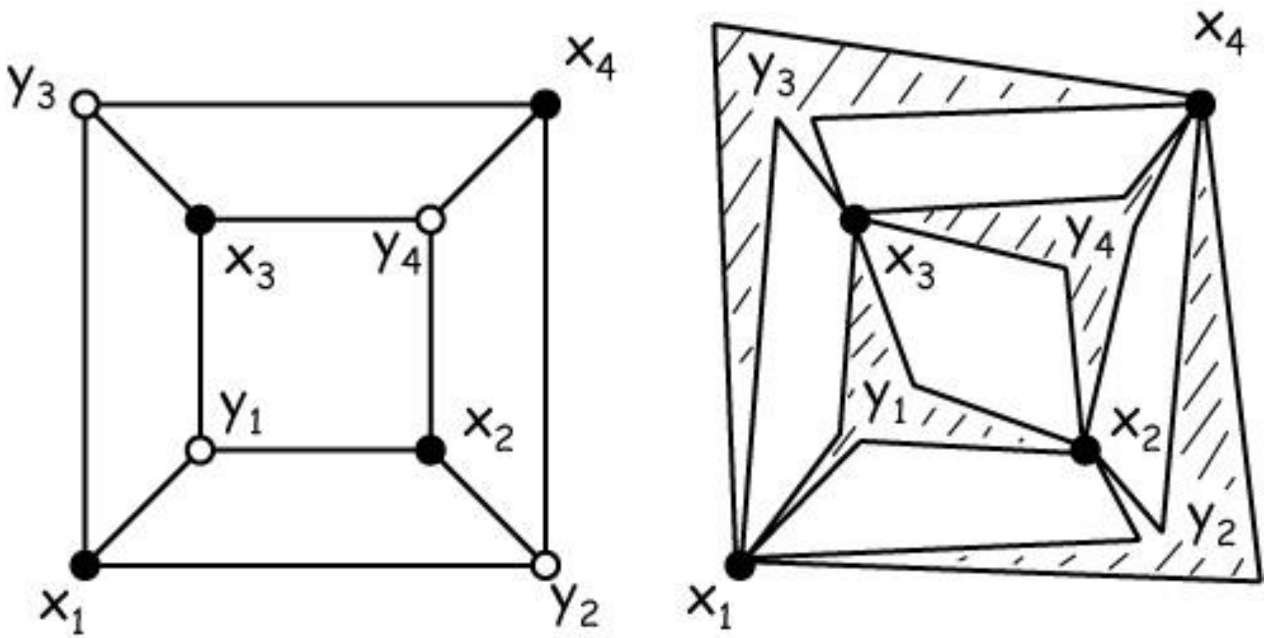
Planar Cover Graph + Height 2

Theorem (Felsner, Li, WTT, '10) If P has height 2 and the cover graph of P is planar, then $\dim(P) \leq 4$.

Fact Both results are best possible as evidenced by S_4.

Key Idea for the Proof

Observation If P has height 2 and the cover graph of P is planar, then P can be considered as the vertex-face poset of a planar multigraph.



Planar Cover Graph + Bounded Height

Theorem (Streib and WTT, '14) For every $h \geq 1$, there is a constant $c(h)$ so that if P has a planar cover graph and the height of P is at most h, then $\dim(P) \leq c(h)$.

Observation The proof uses Ramsey theory at several key places and the bound we obtain for c_h is **very** large in terms of h.

A Key Detail

Observation The cover graph of a poset can be planar and have arbitrarily large tree-width, even when the poset has small height, e.g., consider an $n \times n$ grid.

However The argument used by Streib and WTT used a reduction to the case where the diameter of the cover graph is bounded as a function of the height.

Fact The tree-width of a planar graph of bounded diameter is bounded.

Posets with Outerplanar Cover Graphs

Theorem (Felsner, WTT, Wiechert, '15) If the cover graph of P is outerplanar, then $\dim(P) \leq 4$.

Observation If G is maximal outerplanar, then G has a vertex of degree 2 with both neighbors adjacent in G. It follows easily that the tree-width of G is at most 2.

More Observations on Tree-Width

Observations

- A poset has dimension at most 3 if its cover graph is a tree. Of course, trees have tree-width 1.
- The posets in Kelly's construction have path-width at most 3.
- The Streib-WTT theorem uses a reduction to posets of bounded height. Although planar graphs can have large tree-width, planar graphs of bounded diameter have bounded tree-width.

Bounded Tree-Width

Theorem (Joret, Micek, Milans, WTT, Walczak, Wang, '15+) For every pair (h, t), there is a constant $c(h, t)$ so that if the tree-width of the cover graph of P is at most t and the height of P is at most h, then $\dim(P) \leq c(h, t)$.

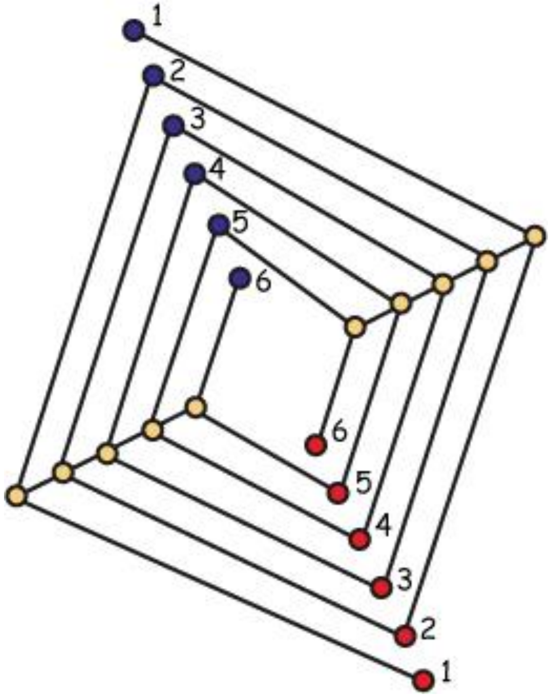



Graph Minors and Bounded Height

Theorem (Walczak, 14+, Micek and Wiechert, '15+) For every pair (n, h), there exists a constant c(n, h) so that if the cover graph of P does not contain a K_n minor and the height of P is at most h, then $\dim(P) \leq c(n, h)$.

Remark Walczak's proof uses deep structural graph theory results. The subsequent proof by Micek and Wiechert is entirely combinatorial. Both proofs are short (less than 10 pages) and very clever.

Revisiting Kelly's Construction



Questions The cover graphs in Kelly's construction have path-width at most 3. Is dimension bounded when path-width is 2? Same question for tree-width 2.

Small Tree and Path-width

Theorem (Biró, Keller and Young '14+) If P is a poset and the path-width of the cover graph of P is 2, then $\dim(P) \leq 17$.

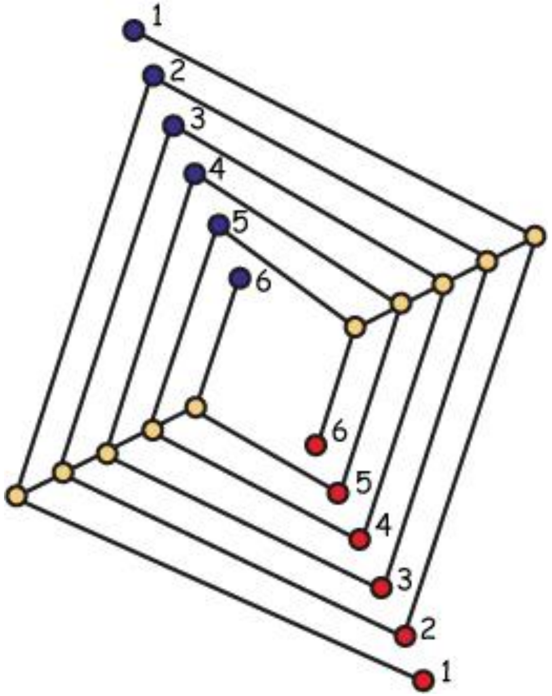
Theorem (Joret, Micek, WTT, Wang and Wiechert, 15+) If P is a poset and the tree-width of the cover graph of P is 2, then $\dim(P) \leq 1276$.

Another Natural Question

Theorem (WTT, Walczak, Wang, 15+) For every d, if $\dim(Q) \leq d$ for every block Q, then $\dim(P) \leq d + 2$.

Remark The inequality is best possible for all d. The case $d = 1$ was done in 1977 and the case $d = 2$ can be extracted from the result on outerplanar graphs. However, when $d \geq 3$, the construction seems to require the product Ramsey theorem and produces incredibly large posets.

Revisiting Kelly's Construction (2)



Question What is the most general notion of sparsity for the cover graph which bounds dimension in terms of height?

Question When the cover graph is planar, is dimension bounded if P excludes two incomparable chains of size k even if the height is not bounded?

The Latest Results

Theorem (Joret, Micek and Wiechert, '16+) Let $\mathcal{C}$ be a class $\mathcal{C}$ of graphs with bounded expansion. Then for every h, there is a constant $c(h)$ so that if the cover graph of P belongs to $\mathcal{C}$ and the height of P is at most h, then $\dim(P) \leq c(h)$.

Theorem (Howard, Streib, WTT, Walczak and Wang, 16+) For every k, there is a constant $c(k)$ so that if the cover graph of P is planar and P excludes $\mathbf{k} + \mathbf{k}$, then $\dim(P) \leq c(k)$.

Remaining Challenges

Conjecture For every pair (n, k), there is a constant c(n, k) so that if P excludes **k** + **k**, and the cover graph of P does not contain a K_n minor, then dim(P) ≤ c(n, k).

Conjecture For every n, there is a constant c(n) so that if P excludes the standard example S_n and the cover graph of P is planar, then dim(P) ≤ c(n).

Conjecture For every pair (n, m), there is a constant c(n, m) so that if P excludes S_n and the cover graph of P does not contain a K_m minor, then dim(P) ≤ c(n, m).