Coefficients of the Laurent expansion of the Hilbert series of Gorenstein rings

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Motivation

Let $R = \bigoplus_{k=0}^{\infty} R_k$ be a finitely generated graded algebra over a field $\mathbb{K} = R_0$. Let

$$\mathrm{Hilb}_R(t) = \sum_{k=0}^{\infty} t^k \dim_{\mathbb{K}} R_k$$

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The Hilbert series converges on $|t| < 1$ and can be expressed in the form

$$\mathrm{Hilb}_R(t) = \frac{h(t)}{(1 - t^{m_1})(1 - t^{m_2}) \cdots (1 - t^{m_s})}, \qquad h(t) \in \mathbb{Z}[t].$$

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If $\dim R > 0$, then $\mathrm{Hilb}_R(t)$ has a pole at $t = 1$ and hence admits a Laurent expansion

$$\mathrm{Hilb}_R(t) = \sum_{k=0}^{\infty} \gamma_k (1 - t)^{k - \dim R} = \frac{\gamma_0}{(1 - t)^{\dim R}} + \frac{\gamma_1}{(1 - t)^{\dim R - 1}} + \cdots$$

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If $G < \mathrm{GL}(V)$ is a finite group and

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$$\gamma_0 = \frac{1}{|G|},$$

and

$$\frac{2\gamma_1}{\gamma_0} = \#\{\text{pseudoreflections in } G\},$$

where a **pseudoreflection** is a $g \in G$ such that V^g has codimension 1.

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Definition *A formal power series* $\varphi(x) = \sum_{k=0}^{\infty}\gamma_k x^k \in \mathbb{C}[\![x]\!]$ *is* **symplectic** *if it satisfies Equation* (1) *for each* $m \geq 1$.

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A rational function $\psi(t)$ *is* **symplectic at** $a \in \mathbb{C}$ **of order** $d \in \mathbb{Z}$ *if*

$$x^d \psi(a - x) \in \mathbb{C}[\![x]\!]$$

is symplectic.

Shift 0

Theorem (Herbig–Herden–S., 2015) *For a formal power series* $\varphi(x) \in \mathbb{C}[\![y]\!] = \sum_{k=0}^{\infty} \gamma_k x^k$, *the following are equivalent.*

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- $\varphi(x) = \varphi\left(\frac{x}{x-1}\right)$.

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Corollary *A rational function $\psi(t)$ is symplectic of order d at $t = a \in \mathbb{C}$ if and only if*

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The **shift** of a Gorenstein algebra R is $r := -(d + a(R))$, hence $d = -a(R)$ corresponds to a shift of 0.

Corollary *A graded Cohen-Macaulay algebra R is Gorenstein with shift 0 if and only if $\mathrm{Hilb}_R(t)$ is symplectic of order $\dim R$ at $t = 1$.*

Alternate Symplectic Basis

The proofs of the above results involve demonstrating that the set

$$\left(\frac{x^2}{1-x}\right)^n, \qquad n \geq 0$$

forms a **symplectic basis** for the algebra of symplectic power series.

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Define $\begin{bmatrix} n \\ k \end{bmatrix}$ *by*

$$x\big(x^{2n} - E_{2n}(x)\big) =: \sum_{k=0}^{2n} \begin{bmatrix} n \\ i \end{bmatrix} x^{2k},$$

i.e. $\begin{bmatrix} n \\ k \end{bmatrix}$ *is the* k*th odd degree coefficient of the* $(2n)$*th Euler polynomial.*

Alternate Symplectic Basis

The first six lines of the triangle of the $\begin{bmatrix} n \\ k \end{bmatrix}$:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & -1 & & 2 & & & & \\
 & & & 3 & & -5 & & 3 & & & \\
 & & -17 & & 28 & & -14 & & 4 & & \\
 & 155 & & -255 & & 126 & & -30 & & 5 & \\
-2073 & & 3410 & & -1683 & & 396 & & -55 & & 6
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Proposition *Define*

$$
\begin{aligned}
\psi_n(x) &:= \frac{1}{(2n-1)!} \sum_{k=0}^{\infty} (-1)^{k-1} E_{k-1}^{(2n-1)}(0)\, x^k \\
&= -x^{2n} - \sum_{k=n}^{\infty} \begin{bmatrix} k \\ n \end{bmatrix} x^{2k+1}.
\end{aligned}
$$

Then the $\psi_n(x)$, $n \geq 0$ form a symplectic basis.

Even Coefficients Determine the Odd (Shift 0)

Theorem (Herbig–Herden–S., 2015) *Let* $\varphi(x) = \sum_{k=0}^{\infty} \gamma_k x^k$ *be a formal power series. Then* $\varphi(x)$ *is symplectic if and only if for each* $n \geq 0$,

$$\gamma_{2n+1} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \gamma_{2k}.$$

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Corollary *For all integers* n, k, ℓ *we have*

$$\begin{bmatrix} n-k \\ \ell \end{bmatrix} + \begin{bmatrix} n-\ell \\ k \end{bmatrix} = \begin{bmatrix} n \\ k+\ell \end{bmatrix} + \sum_i \sum_r \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} r-1 \\ k \end{bmatrix} \begin{bmatrix} i-r \\ \ell \end{bmatrix}.$$

Shift $r \neq 0$

Recall the Gorenstein condition

$$\mathrm{Hilb}_R(1/t) = (-1)^d t^{-a(R)} \mathrm{Hilb}_R(t).$$

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For the power series $\varphi(x) := x^d \mathrm{Hilb}_R(1-x)$, this corresponds to

$$\varphi\left(\frac{x}{x-1}\right) = (1-x)^r \varphi(x).$$

Shift $r > 0$

Theorem (Herbig–Herden–S.) *A power series* $\varphi(x) = \sum_{k=0}^{\infty} \gamma_k x^d$ *satisfies*

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for some $r > 0$ *if and only if, for each* $m \geq 1$,

$$\sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} \gamma_{m+i-r} = 0, \quad \textbf{if } \boldsymbol{r} \textbf{ is even,}$$

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Shift $r = 1$

1	**2**	0									
0	**3**	**9**	**6**	0							
0	0	**10**	**40**	**50**	**20**	0					
0	0	0	**35**	**175**	**315**	**245**	**70**	0			
0	0	0	0	**126**	**756**	**1764**	**2016**	**1134**	**252**	0	

$\cdots$

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0	**3**	**9**	**6**	0						
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0	0	0	0	**126**	**756**	**1764**	**2016**	**1134**	**252**	0

...

Dividing by common factors in each row, the resulting nonzero coefficients are the **Lucas triangle:**

$$\begin{array}{ccccccccccc} & & & & 1 & & 2 & & & & \\ & & & 1 & & 3 & & 2 & & & \\ & & 1 & & 4 & & 5 & & 2 & & \\ & 1 & & 5 & & 9 & & 7 & & 2 & \\ 1 & & 6 & & 14 & & 16 & & 9 & & 2 \end{array}$$

...

Shift $r > 0$ odd

$\boldsymbol{r = 3:}$

3	**2**	0								
0	**10**	**15**	**6**	0						
0	0	**35**	**84**	**70**	**20**	0				
0	0	0	**126**	**420**	**540**	**315**	**70**	0		
0	0	0	0	**462**	**1980**	**3465**	**3080**	**1386**	**252**	0

$\cdots$

Shift $r > 0$ odd

$r = 3$:

3	**2**	0								
0	**10**	**15**	**6**	0						
0	0	**35**	**84**	**70**	**20**	0				
0	0	0	**126**	**420**	**540**	**315**	**70**	0		
0	0	0	0	**462**	**1980**	**3465**	**3080**	**1386**	**252**	0
						...				

$r = 5$:

5	**2**	0								
0	**21**	**21**	**6**	0						
0	0	**84**	**144**	**90**	**20**	0				
0	0	0	**330**	**825**	**825**	**385**	**70**	0		
0	0	0	0	**1287**	**4290**	**6006**	**4368**	**1638**	**252**	0
						...				

Shift $r < 0$

Theorem (Herbig–Herden–S.) *A power series* $\varphi(x) = \sum_{k=0}^{\infty} \gamma_k x^d$ *satisfies*

$$\varphi\left(\frac{x}{x-1}\right) = (1-x)^r \varphi(x)$$

for some $r < 0$ *if and only if, for* $1 \leq m \leq \lceil -r/2 \rceil$*,*

$$\sum_{i=0}^{m} \frac{(1-r-m-i)!\,(m+i)!}{(m-i)!\,i!} \gamma_{m+i-1} = 0$$

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Shift $r = -6$

$$
\begin{array}{ccccccccccccc}
\mathbf{720} & \mathbf{-240} & 0 & & & & & & & & & & \\
0 & \mathbf{120} & \mathbf{-144} & \mathbf{72} & 0 & & & & & & & & \\
0 & 0 & \mathbf{24} & \mathbf{-72} & \mathbf{120} & \mathbf{-120} & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 \\
 & & & & & & \cdots & & & & & &
\end{array}
$$

Or:

$$
\begin{array}{ccccccccccccc}
\mathbf{3} & \mathbf{-1} & 0 & & & & & & & & & & \\
0 & \mathbf{5} & \mathbf{-6} & \mathbf{3} & 0 & & & & & & & & \\
0 & 0 & \mathbf{1} & \mathbf{-3} & \mathbf{5} & \mathbf{-5} & 0 & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & \mathbf{1} & 0 \\
 & & & & & & \cdots & & & & & &
\end{array}
$$

Shift $r = -5$

$$
\begin{array}{cccccccccccc}
\mathbf{120} & \mathbf{-48} & 0 & & & & & & & & & \\
0 & \mathbf{24} & \mathbf{-36} & \mathbf{24} & 0 & & & & & & & \\
0 & 0 & \mathbf{6} & \mathbf{-24} & \mathbf{60} & \mathbf{-120} & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{3} & \mathbf{9} & \mathbf{6} & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{10} & \mathbf{40} & \mathbf{50} & \mathbf{20} & 0 \\
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\end{array}
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$$
\begin{array}{cccccccccccc}
\mathbf{5} & \mathbf{-2} & 0 & & & & & & & & & \\
0 & \mathbf{2} & \mathbf{-3} & \mathbf{2} & 0 & & & & & & & \\
0 & 0 & \mathbf{1} & \mathbf{-4} & \mathbf{10} & \mathbf{-20} & 0 & & & & & \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{2} & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{3} & \mathbf{2} & 0 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{4} & \mathbf{5} & \mathbf{2} & 0 \\
 & & & & & & \cdots & & & & &
\end{array}
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Hence $\mathcal{F} = \bigoplus_{r\in\mathbb{Z}} \mathcal{F}_r$ is a $\mathbb{Z}$-graded algebra generated by $x^2/(1-x)$ and $x-2$.

Shift $r \neq 0$: Even Coefficients Determine the Odd

Definition *Define* $\left\{ {n \atop k} \right\}$ *by*

$$x\left(x^{2n+1} - E_{2n+1}(x)\right) =: \sum_{k=0}^{2n+1} \left\{ {n \atop k} \right\} x^{2k},$$

i.e. $\left\{ {n \atop k} \right\}$ *is the* k*th even degree coefficient of the* $(2n+1)$*st Euler polynomial.*

Shift $r \neq 0$: Even Coefficients Determine the Odd

Definition *Define* $\left\{ {n \atop k} \right\}$ *by*

$$x\left(x^{2n+1} - E_{2n+1}(x)\right) =: \sum_{k=0}^{2n+1} \left\{ {n \atop k} \right\} x^{2k},$$

i.e. $\left\{ {n \atop k} \right\}$ *is the* k*th even degree coefficient of the* $(2n+1)$*st Euler polynomial.*

The first six lines of the triangle of the $\left\{ {n \atop k} \right\}$:

$$\frac{1}{2}$$

$$-\frac{1}{4} \qquad \frac{3}{2}$$

$$\frac{1}{2} \qquad -\frac{5}{2} \qquad \frac{5}{2}$$

$$-\frac{17}{8} \qquad \frac{21}{2} \qquad -\frac{35}{4} \qquad \frac{7}{2}$$

$$\frac{31}{2} \qquad -\frac{153}{2} \qquad 63 \qquad -21 \qquad \frac{9}{2}$$

$$-\frac{691}{4} \qquad \frac{1705}{2} \qquad -\frac{2805}{4} \qquad 231 \qquad -\frac{165}{4} \qquad \frac{11}{2}$$

Theorem (Herbig–Herden–S.) $\varphi(x) = \sum_{k=0}^{\infty} \gamma_k x^k \in \mathcal{F}_r$ *if and only if:*

$$\gamma_{2n+1} = \sum_{k=0}^{n} \left\{ {n+m \atop k+m} \right\} \gamma_{2k} \qquad n \geq 0, \qquad \boldsymbol{r = 2m + 1 > 0}$$

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Thank you!