Towards an excluded-minor characterization of the Hydra-5 matroids

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Matroids

- Matroid theory is the study of dependence.
- Some matroids come from vector spaces or graphs, but some do not.

Matroids

An example

- Consider the following matrix over the field $\mathrm{GF}(5)$.

$$
\begin{array}{ccccc}
a & b & c & d & e \\
\end{array}
$$
$$
\begin{bmatrix}
1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 2 & 3
\end{bmatrix}
$$

- Let $\mathcal{I}$ be the collection of linearly independent subsets of $\{a, b, c, d, e\}$.
- Then $(\{a, b, c, d, e\}, \mathcal{I})$ is a matroid, $U_{2,5}$.

A geometric representation of the matroid $U_{2,5}$

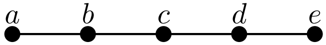
- The matroid $U_{2,5}$ is said to be *represented* by the following matrix over the field $\mathrm{GF}(5)$.

$$\begin{array}{c} \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \end{bmatrix} \end{array}$$

  which is more compactly written as

$$\begin{array}{cc} & \begin{matrix} c & d & e \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \end{array}$$

- A geometric representation of the matroid $U_{2,5}$ is the 5-point line.

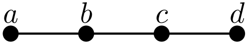


Characterizing matroids with matrix representations

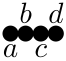
- **(Whitney, 1935)** "The fundamental question of completely characterizing systems which represent matrices is left unsolved."

Minors

- **Deletion**: $U_{2,5} \backslash e$



- **Contraction**: $U_{2,5}/e$



- **Minor**: Sequence of deletions and contractions.

Minor order

- The class of $\mathbb{F}$-representable matroids is minor-closed.

Minors

Deletion

- $U_{2,5}$ with $\mathrm{GF}(5)$-representation:

$$\begin{array}{c c} & \begin{array}{ccc} c & d & e \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \end{array}$$

- $U_{2,5} \backslash e$: remove the column e.

$$\begin{array}{c c} & \begin{array}{cc} c & d \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \end{array}$$

Minors

Contraction

- $U_{2,5}$ with $\mathrm{GF}(5)$-representation:

$$\begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} c & d & e \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 2 & 3 \end{array}\right] \end{array}$$

- $U_{2,5}/e$: change basis to $\{b, e\}$ by ‘pivoting’.

$$\begin{array}{c} \\ e \\ b \end{array} \begin{array}{c} \begin{array}{ccc} a & c & d \end{array} \\ \left[\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 3 & 4 \end{array}\right] \end{array}$$

- then remove the row e.

$$\begin{array}{c} \\ b \end{array} \begin{array}{c} \begin{array}{ccc} a & c & d \end{array} \\ \left[\begin{array}{ccc} 2 & 3 & 4 \end{array}\right] \end{array}$$

Excluded minors

A matroid M is an **excluded minor** for a minor-closed class $\mathcal{C}$ if:

- $M \notin \mathcal{C}$; and
- Any minor of M is in $\mathcal{C}$.

Excluded-minor characterizations

- **(Tutte, 1958)** *One* excluded minor for $\mathrm{GF}(2)$-representability.
- **(Bixby, 1979; Seymour, 1979)** *Four* excluded minors for $\mathrm{GF}(3)$-representability.
- **(Geelen, Gerards, & Kapoor, 2000)** *Seven* excluded minors for $\mathrm{GF}(4)$-representability.

$\mathrm{GF}(5)$-representable matroids

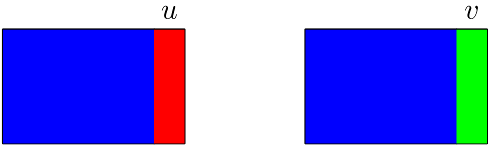
(Mayhew & Royle, 2008) *At least 564* excluded minors for $\mathrm{GF}(5)$-representability.

Inequivalent representations

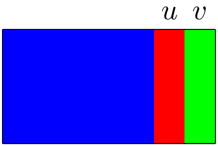
(Oxley, Vertigan, & Whittle, 1996) Up to six inequivalent representations over $\mathrm{GF}(5)$.

Deletion pair u, v

Use representations of $M\backslash u$ and $M\backslash v$



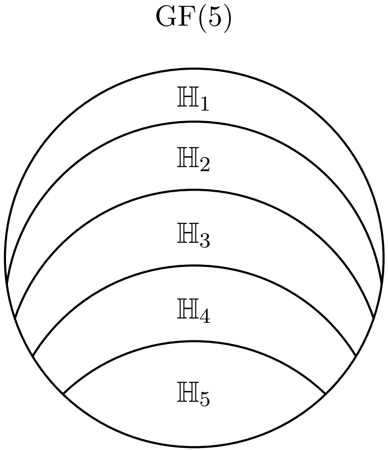
To build a candidate representation for M



$\mathbb{H}_k$-representable matroids
(Pendavingh and Van Zwam, 2010)

- $\mathbb{H}_k$-representable matroids are the $\mathrm{GF}(5)$-representable matroids with at least k inequivalent representations.

GF(5)-representable matroids



$\mathbb{H}_5$-**representable**: GF(5)-representable matroids with at least five inequivalent representations.

Unique representability

Unique representability in $\mathbb{H}_5$ with a $U_{2,5}$-minor and enough connectivity.

Gaining traction on the $\mathrm{GF}(5)$ problem

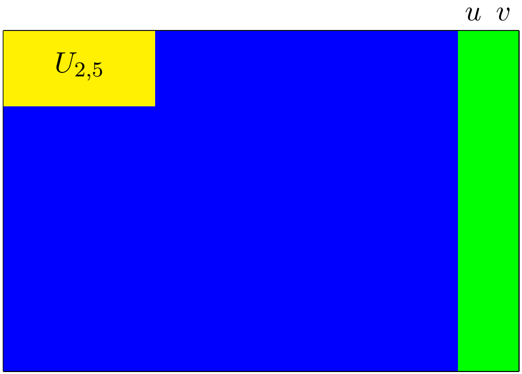
If an excluded minor M for $\mathbb{H}_5$ is $\mathbb{H}_4$-representable, then we recover unique representability in $\mathbb{H}_4$ with an M-minor and enough connectivity.

Goal

- Find all of the excluded minors for the class of Hydra-5 matroids.

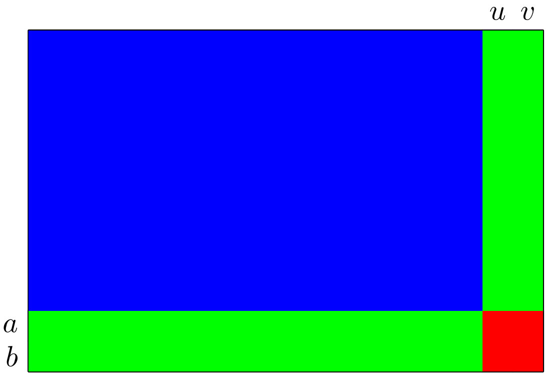
Another look at the candidate representation

Excluded minor M for $\mathbb{H}_5$ with a $U_{2,5}$-minor.

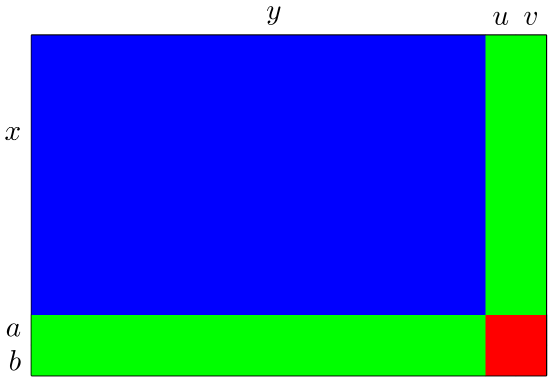


Incriminating set $\{a, b, u, v\}$

Certificate of non-representability.

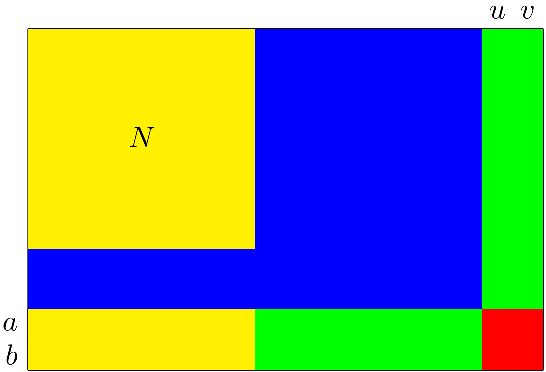


What about the other elements?



We lose the $U_{2,5}$-minor or lose connectivity by removing x or y.

$U_{2,5}$-fragile minor



For all $e \in N$, either $N \backslash e$ or N/e has no $U_{2,5}$-minor.

To bound the excluded minor M

We need to know

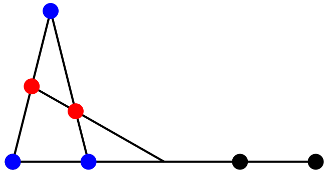
- How big is the $U_{2,5}$-fragile minor?
- How many additional elements are needed to repair connectivity?

A natural question

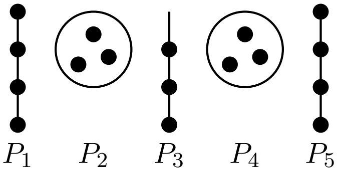
What is the structure of $\mathbb{H}_5$-representable $U_{2,5}$-fragile matroids?

Fan extensions

A fan extension of $U_{2,5}$:



Path extensions



(Oxley, Semple, & Vertigan, 2000) Generalized Δ-Y exchange.

Theorem

(Clark, Mayhew, Whittle, & Van Zwam, 2015)

The following statements are equivalent for a matroid M.

(i) M is a $\{U_{2,5}, U_{3,5}\}$-fragile $\mathbb{H}_5$-representable matroid.

(ii)
  (a) $|M| \leq 9$; or
  (b) M is a member of a fan family; or
  (c) M is a member of the path family.

To bound the excluded minor M

We need to know

- How big is the $U_{2,5}$-fragile minor?
- How many additional elements are needed to repair connectivity?

Theorem

(Clark, Oxley, Semple & Whittle, 2015)

Let M be an excluded minor for the class of $\mathbb{H}_5$-representable matroids. If M has a pair of elements u, v such that $M\backslash u, v$ is 3-connected with a $U_{2,5}$-minor, then at least one of the following holds.

(i) M is "small"; or

(ii) $M\backslash u, v$ is $U_{2,5}$-fragile.

Thank you!