Decomposing Cayley Digraphs into Isomorphic Directed Trees

3rd Mississippi Discrete Mathematics Workshop

Erik Westlund
joint work Mari Castle and Evan Moore

Department of Mathematics
Kennesaw State University

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Preliminaries

Definition

Decomposition of $X = (V, E)$ is $\{X_1, \ldots, X_b\}$ where

1. $E(X) = E(X_1) \cup \cdots \cup E(X_b)$, and
2. $E(X_i) \cap E(X_j) = \varnothing$ for all $i \neq j$.

Definition

(di)graphs H and X, an H-**decomposition** of X is a decomposition $\{X_1, \ldots, X_b\}$ such that $X_i \cong H$ for all i.

History

Conjecture (Ringel 1963)

If T is any tree with m edges, then K_{2m+1} has a T-decomposition.

Conjecture (Graham-Häggkvist 1984)

If T is any tree with m edges, then every $2m$-regular graph and every m-regular bipartite graph has a T-decomposition.

Known results

Theorem (Snevily 1991)

Let X be a $2m$-regular graph with girth g and let T be a tree with m edges and diameter d. If $g > d$, then X has a T-decomposition.

Theorem (Snevily)

If T is a tree with m edges, and X is Cartesian product of m cycles, then X is T-decomposable.

Theorem (Kouider-Lonc 1999)

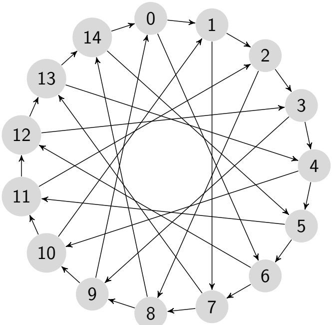
For any $m \leq 2g - 3$, every $2m$-regular graph with girth at least g decomposes into paths of length m.

Cayley Digraphs

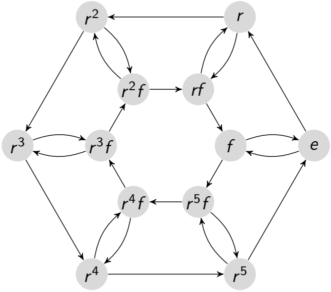
Definition

G finite group; $S \subseteq G - \{e\}$, the **Cayley digraph** $\boldsymbol{X = \overrightarrow{\mathrm{Cay}}(G;S)}$ of G with connection set S:

- $V(X) = G$
- $E(X) = \{(x, xs) : s \in S\}$



$\overrightarrow{\mathrm{Cay}}(\mathbb{Z}_{15}; \{1, 6\})$



$\overrightarrow{\mathrm{Cay}}(D_6; \{r, f\})$

Why Cayley Digraphs?

- Lots of symmetry and algebraic structure.
- Large literature on decomposition problems.
- Cayley graphs and digraphs are regular ($\overrightarrow{\text{Cay}}(G; S)$ is $|S|$-regular).
- Characterized by regular subgroup of automorphisms (Sabidussi).

Remark (Sabidussi 1958)

If $g \in G$, then the permutation given by

$$\phi_g : v \mapsto gv, g \in G$$

is a graph automorphism of $\overrightarrow{\text{Cay}}(G; S)$. $G_L = \{\phi_g : g \in G\} \leq \text{Aut}(\overrightarrow{\text{Cay}}(G; S))$.

Previous Results

Theorem (El-Zanati, Vanden Eynden, Stubbs 2000)

Let G be a finite group $|G : H| = 2$. If $S \subset G - H$ is square-independent, containing n_1 non-involutions and n_2 involutions, and T is any tree with $2n_1 + n_2$ edges, then $Cay(G; S)$ has a T-decomposition into $|G|/2$ copies of T.

Theorem (Fink 1994)

If T is any directed tree with m arcs, and G is a group with minimal generating set S, where $|S| = m$, then $\overrightarrow{Cay}(G; S)$ is T-decomposable.

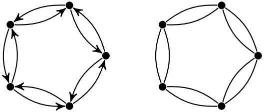
Directed version: If T is a directed tree with m arcs, and X a directed simple m-regular graph, what are sufficient conditions for X to be T-decomposable?

Directed Graham-Häggkvist

Theorem (Colbourn, Hoffman, Rodger 1991)

X symmetric directed simple m-regular graph, $m = \alpha + \beta$. There exists a (α, β)-directed star decomposition of X $\Leftrightarrow$ underlying graph of X has an α-factor.

Any 2-regular symmetric digraph of odd order has no $(1, 1)$-DSD. (No 1-factor.)



Corollary

S inverse-closed, $|S| = \alpha + \beta$, then $X = DCay(G; S)$ has an (α, β)-DSD $\Leftrightarrow$ the underlying graph of X has an α-factor.

Relaxing Minimality

Definition

An S-**word**, where $S = \{s_1, \ldots, s_m\} \subseteq G$, is finite product of distinct elements:

$$s_{\sigma(1)}^{n_1} s_{\sigma(2)}^{n_2} \cdots s_{\sigma(m)}^{n_m}$$

$n_i \in \{-1, 0, 1\}$ and $\sigma \in \mathrm{SYM}(m)$. The **length** $\ell_S(g)$ of $g \in G$ is the min number of nonzero n_i's needed to express g as an S-word. If g is not an S-word, then $\ell_S(g) = \infty$.

$G = S_4$ and $S = \{s_1 = (12), s_2 = (13)(24), s_3 = (1234)\}$, then

$$g = (134) = s_3^{-1} s_2 s_1 \Rightarrow \ell_S(g) \leq 3$$

but

$$g = (134) = s_3 s_1 \Rightarrow \ell_S(g) \leq 2$$

and $g, g^{-1} = (143) \notin S$, so $\ell_S(g) = 2$

Relaxing Minimality

Definition

For $0 \leq k, t \leq |S|$, a set S is (k,t)-**word degenerate** if S contains exactly t elements, $\{s_1, \ldots, s_t\}$ such that for each i, $\ell_U(s_i) < k$ and $U = S - \{s_i\}$.

- $G = S_8; S = \{s_1, \ldots, s_7\}$, where $s_1 = (1234)$, $s_2 = (12)$, $s_3 = (134)$, $s_4 = s_1^2$, $s_5 = (452)$, $s_6 = (153)$, $s_7 = (67)$
- SAGE: $\ell(s_1) = \ell(s_2) = \ell(s_3) = \ell(s_5) = 2$, $\ell(s_4) = 3$, $\ell(s_6) = 4$, $\ell(s_7) = \infty$
- S is $(4,5)$-word degenerate set
- S is also $(3,4)$- and $(5,6)$-word degenerate.

We can now control the number of elements with "small length."

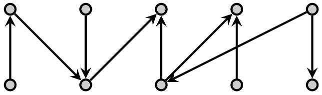
Minimal Spanning Star Forests

Definition

Directed **spanning star forest** (SSF) is spanning subgraph with directed star components. A SSF of T is *minimal*, if it contains min # of components.

Proposition

Any minimal SSF of a directed tree T contains the maximum number of arcs out of all directed spanning star forests of T.



Directed Tree with 9 arcs.

Idea: take minimal SSF of T, saturate arcs of stars with degenerate elements, to keep "bad" elements in local star communities and "break up" the effects of non-minimality on vertex-labeling. No degenerate-labeled 3-semipaths.

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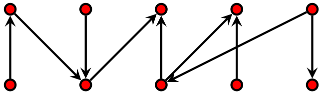

SSF with 10 components, 0 arcs.

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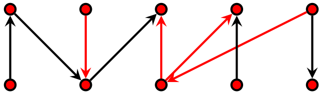

SSF with 6 components, 4 arcs.

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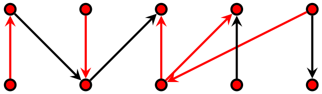

SSF with 5 components, 5 arcs.

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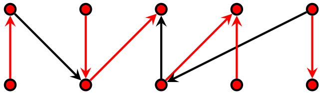

MSSF with 4 components, 6 arcs.

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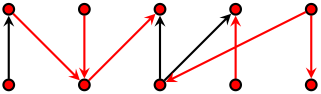

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Main Result

Theorem (Castle, Moore, W.)

If T is any directed tree with m arcs, minimal SSF N, and S is a (k,t)-word degenerate m-subset of G where $k \geq \operatorname{diam}(T)$ and $t \leq |E(N)|$, then $X = \overrightarrow{Cay}(G;S)$ is T-decomposable unless the following are all true:

1. *S is inverse-closed and involution-free;*
2. *T is an $(\alpha, m-\alpha)$-directed star;*
3. *Underlying graph of X does not have an α-factor.*

Corollaries

Corollary

If S is any (k,t)-word degenerate m-subset of a group G, and T is any directed tree having m arcs and a minimal SSF N, then $\overrightarrow{\text{Cay}}(G;S)$ is T-decomposable whenever $k \geq \text{diam}(T) \geq 3$, and $t \leq |E(N)|$.

Minimal generating set $S \Rightarrow (|S|,0)$-word degenerate.

Corollary (Fink)

Let G be a group with minimal generating set S. If T is any directed tree with $|S|$ arcs, then $\overrightarrow{\text{Cay}}(G;S)$ is T-decomposable.

Square-independent set $S \Rightarrow (|S|,0)$-word degenerate.

Corollary (Analogue to El-Zanati et. al)

If S is a square-independent m-subset of a finite group G, and T is any directed tree with m arcs, then $\overrightarrow{\text{Cay}}(G;S)$ is T-decomposable.

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Main Result Proof Sketch

Proof.

- Label $E(F)$ with degenerate elements of S, remaining $E(T)$ with remaining elements of S, and one (root) vertex with e.
- $S_v = [s_{i_1}, s_{i_2}, \ldots, s_{i_d}]$ is arc-label sequence on semipath from root to v.
- Label v with $\ell(v) = s_{i_1}^{\pm 1} s_{i_2}^{\pm 1} \ldots s_{i_d}^{\pm 1}$, depending on arc direction.
- If $\ell(u) = \ell(v)$, then all vertices on semi-path from u to v must be expressable as words of length $< \mathrm{diam}(T)$, so are degenerate.
- t degenerate elements all lie in directed stars, so cannot produce semipaths of length 3 or more
- only possibility is degenerate-labeled directed 2-paths must exist
- Permute degenerate element arc labels to "fix."
- T is viewed as a subgraph of $\overrightarrow{\mathrm{Cay}}(G; S)$.
- Take orbit of T under action of Sabidussi automorphisms.

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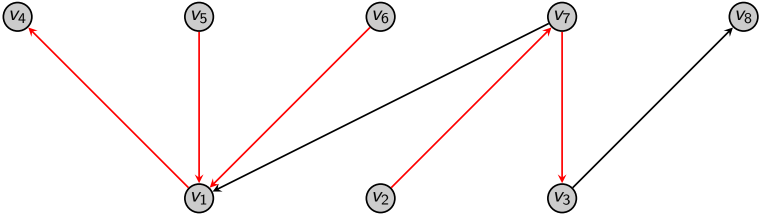
Step 1 - Choose a Tree

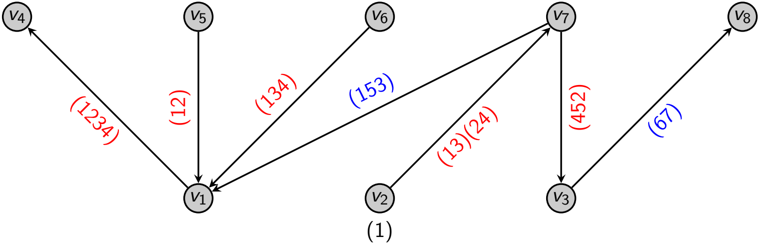
- $X = \overrightarrow{\text{Cay}}(S_8; \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\})$
- $s_1 = (1234)$, $s_2 = (12)$, $s_3 = (134)$, $s_4 = s_1^2$, $s_5 = (452)$, $s_6 = (153)$, $s_7 = (67)$
- S is $(4, 5)$-word degenerate set
- $k = 4 \geq \text{diam}(T) = 4$ and $t = 5 \leq |E(F)| = 5$



T

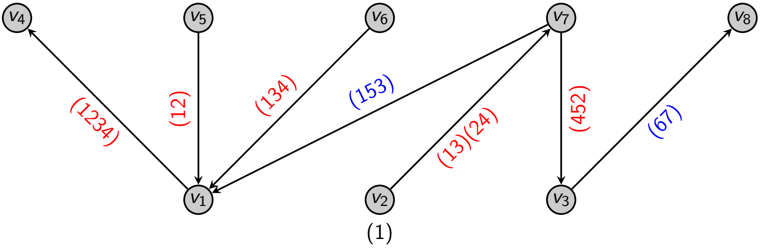
Step 2 - Choose Root and Label Edges

- $X = \overrightarrow{\text{Cay}}(S_8; \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\})$
- $s_1 = (1234)$, $s_2 = (12)$, $s_3 = (134)$, $s_4 = s_1^2$, $s_5 = (452)$, $s_6 = (153)$, $s_7 = (67)$



Step 3 - Label Vertices

- $X = \overrightarrow{\text{Cay}}(S_8; \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\})$
- $s_1 = (1234)$, $s_2 = (12)$, $s_3 = (134)$, $s_4 = s_1^2$, $s_5 = (452)$, $s_6 = (153)$, $s_7 = (67)$
- $S_{v_5} = [s_4, s_6, s_2]$



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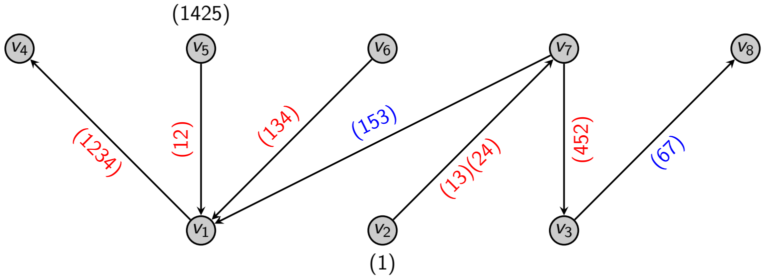
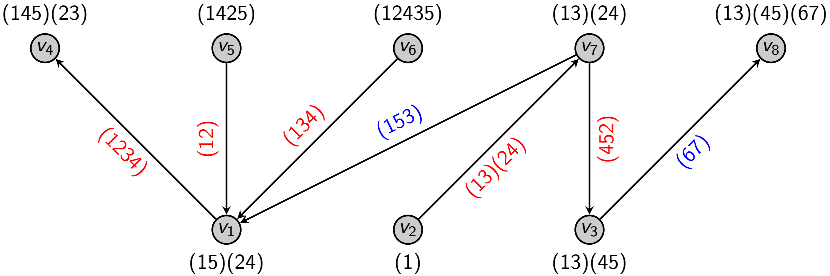
- $\ell(v_5) = s_4 s_6 s_2^{-1} = (13)(24)(153)(12) = (1425)$



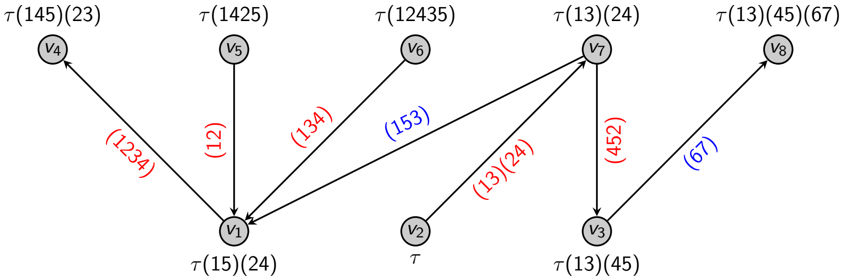
Step 3 - Label Vertices

T is now a subgraph of $\overrightarrow{\mathrm{Cay}}(S_8; \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\})$



Step 4 - Apply Automorphisms

Apply elements of $G_L = \{\phi_\tau : \tau \in S_8\}$, where $\phi_\tau : \sigma \mapsto \tau\sigma$



$$(x, y) \in E \Leftrightarrow x^{-1}y = s \in S$$

$$\Leftrightarrow x^{-1}\tau^{-1}\tau y = (\tau x)^{-1}(\tau y) = s$$

$$\Leftrightarrow (\tau x, \tau y) \in E$$

Thank you!