Multiplicative Zagreb indices of k-trees

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Directed by Bing Wei

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  - k-trees
  - Zagreb Index and Multiplicative Zagreb Index

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- Our results

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  - k-trees
  - Zagreb Index and Multiplicative Zagreb Index
- Our results
- Main proofs

k-trees

Definition (Beineke and Pippert 1969)

The k-**tree**, denoted by T_n^k, for positive integers n, k with $n \geq k$, is defined recursively as follows:
The smallest k-tree is the k-clique K_k. If G is a k-tree with $n \geq k$ vertices and a new vertex v of degree k is added and joined to the vertices of a k-clique in G, then the obtained graph is a k-tree with $n+1$ vertices.

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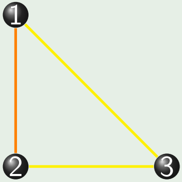

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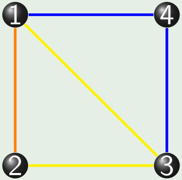

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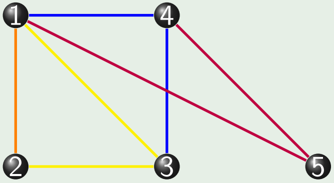

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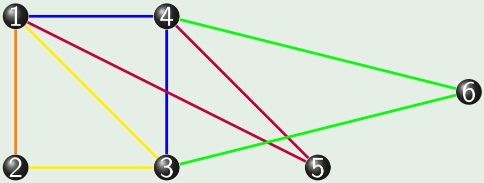

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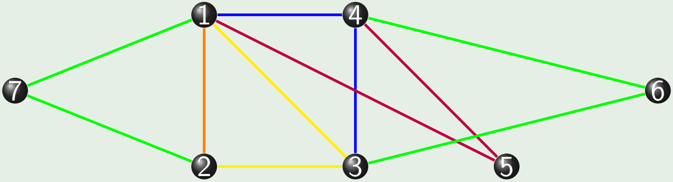

k-simplicial vertex

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A vertex $v \in V(T_n^k)$ is called a k-simplicial vertex if v is a vertex of degree k whose neighbors form a k-clique of T_n^k.

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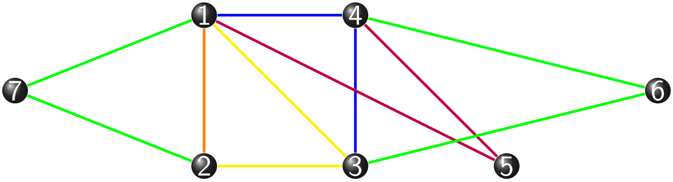
- In the following 2-tree, $5, 6, 7$ are 2-simplicial vertices.



k-simplicial vertex

- Let $S_1(T_n^k)$ be the set of all simplicial vertices of T_n^k, for $n \geq k+2$, and set $S_1(K_k) = \phi, S_1(K_{k+1}) = \{v\}$, where v is any vertex of K_{k+1}.

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- Let $G = G_0, G_i = G_{i-1} - v_i$, where v_i is a simplicial vertex of G_{i-1}, then $\{v_1, v_2 ... v_n\}$ is called **a simplicial elimination ordering** of the n-vertex graph G.

- The k-**path**, denoted by P_n^k, for positive integers n, k with $n \geq k$, is defined as follows:
  Starting with a k-clique $G[\{v_1, v_2...v_k\}]$. For $i \in [k+1, n]$, the vertex v_i is adjacent to vertices $\{v_{i-1}, v_{i-2}...v_{i-k}\}$ only.

k-path and k-star

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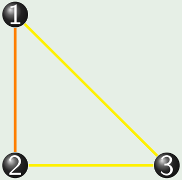

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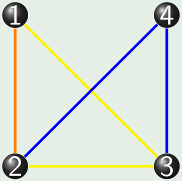

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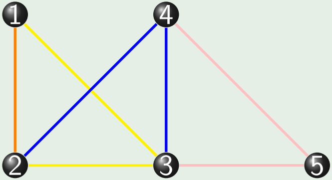

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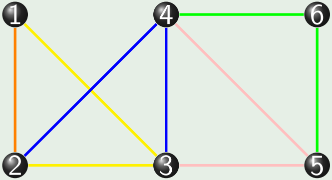

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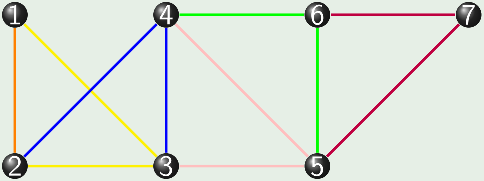

- The k-**star**, denoted by $S_{k,n-k}$, for positive integers n, k with $n \geq k$, is defined as follows:
  Starting with a k-clique $G[\{v_1, v_2 ... v_k\}]$ and an independent set S with $|S| = n - k$. For $i \in [k+1, n]$, the vertex v_i is adjacent to vertices $\{v_1, v_2 ... v_k\}$ only.

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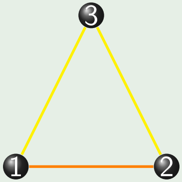

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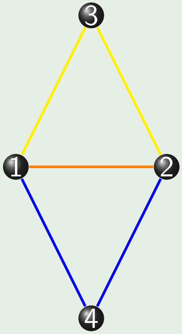

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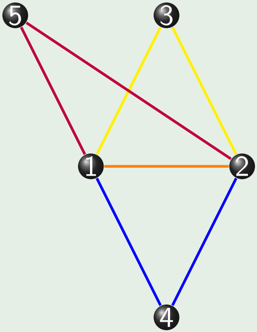

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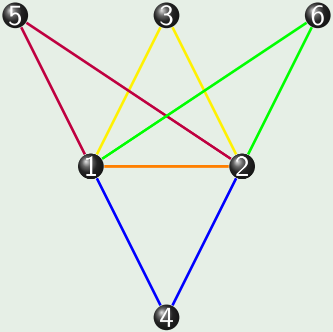

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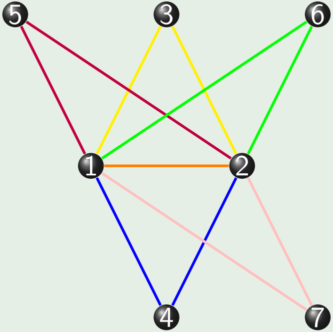

Hyper pendent edge

Definition

If $w(G-S) \leq 2$ for any k-clique $G[S]$ of T_n^k, we say T_n^k is a hyper pendent edge; If there exists a k-clique $G[S]$ with $w(G-S) \geq 3$, let C be a component of $T_n^k - S$ and contain a unique vertex belonging to $S_1(G)$, then we say that $G[V(S) \cup V(C)]$ is a hyper pendent edge of T_n^k, denoted by $\mathcal{P}$.

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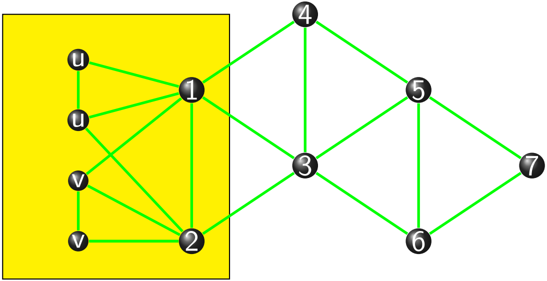
- 2-tree: $S = \{1, 2\}$, $C = \{3, 4, 5, 6, 7\}$



Definition

The first and second **Zagreb indices** of the graph $G = (V, E)$ are defined as

$$M_1 = \sum_{v \in V(G)} d(v)^2; M_2 = \sum_{uv \in E(G)} d(u)d(v).$$

Zagreb Indices

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- Let P_n, S_n be the path and star on n vertices, respectively, then

$$M_1(P_n) = 4n - 6, M_1(S_n) = n^2 - n;$$

$$M_2(P_n) = 4n - 8, M_2(S_n) = n^2 - 2n + 1.$$

Theorem (Das and Gutman 2004)

Let T be any tree on n vertices, then

$$M_1(P_n) \leq M_1(T) \leq M_1(S_n)$$

$$M_2(P_n) \leq M_2(T) \leq M_2(S_n)$$

the left-side and the right-side equalities are reached if and only if $T \cong P_n$ and $T \cong S_n$, respectively.

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Theorem (Estes and Wei 2012)

Let T_n^k be any k-tree on n vertices, then

$$M_1(P_n^k) \leq M_1(T_n^k) \leq M_1(S_{k,n-k})$$

$$M_2(P_n^k) \leq M_2(T_n^k) \leq M_2(S_{k,n-k})$$

the left-side and the right-side equalities are reached if and only if $T_n^k \cong P_n^k$ and $T_n^k \cong S_{k,n-k}$, respectively.

Definition (Todeschini, Ballabio, Consonni 2010)

The first and second **Multiplicative Zagreb indices** of the graph $G = (V, E)$ are defined as

$$\prod_1(G) = \prod_{v \in V(G)} d(v)^2; \prod_2(G) = \prod_{uv \in E(G)} d(u)d(v).$$

Multiplicative Zagreb Indices

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Theorem (Gutman 2011)

Let $n \geq 5$ and T_n be any tree with n vertices, then

$$\prod_1(S_n) \leq \prod_1(T_n) \leq \prod_1(P_n);$$

$$\prod_2(P_n) \leq \prod_2(T_n) \leq \prod_2(S_n).$$

Definition

The first **generalized** and second **Multiplicative Zagreb indices** of graph $G = (V, E)$ are defined as follows: for any real number $c > 0$,

$$\prod_{1,c}(G) = \prod_{v \in V(G)} d(v)^c;$$

$$\prod_{2}(G) = \prod_{uv \in E(G)} d(u)d(v) = \prod_{v \in V(G)} d(v)^{d(v)}.$$

Our results

Theorem

Let T_n^k be a k-tree on $n \geq k$ vertices, then

$$(1) \prod_{1,c}(S_{k,n-k}) \leq \prod_{1,c}(T_n^k) \leq \prod_{1,c}(P_n^k)$$

$$(2) \prod_{2}(P_n^k) \leq \prod_{2}(T_n^k) \leq \prod_{2}(S_{k,n-k})$$

For (1), the left-side and the right-side equalities are reached if and only if $T_n^k \cong S_{k,n-k}$ and $T_n^k \cong P_n^k$, respectively; For (2), the left-side and the right-side equalities are reached if and only if $T_n^k \cong P_n^k$ and $T_n^k \cong S_{k,n-k}$, respectively.

Analytic Tool

- The function $f(x) = \dfrac{x}{x+m}$ is strictly increasing for $x \in [0, \infty)$, where m is a positive integer.
- The function $g(x) = \dfrac{x^x}{(x+m)^{x+m}}$ is strictly decreasing for $x \in [0, \infty)$, where m is a positive integer.

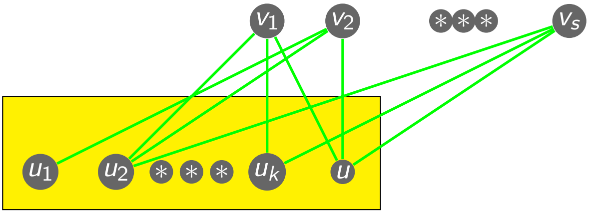
Sketch of the Proof

- We first show that $\prod_{1,c}(T_n^k) \geq \prod_{1,c}(S_{k,n-k})$, $\prod_2(T_n^k) \leq \prod_2(S_{k,n-k})$, it suffices to prove the following lemma.

Lemma (1)

For any k-tree $G \ncong S_{k,n-k}$, let $u \in S_2$, $N(u) \cap S_1 = \{v_1, v_2 ... v_s\}$, where $s \geq 1$ is an integer, then
(i) For any i with $1 \leq i \leq s$, there exists a vertex $v \in N(u) - \{v_1, v_2 ... v_s\}$ of degree at least k in $G[V(G) - \{v_1, v_2 ... v_s\}]$ such that $vv_i \notin E(G)$.
(ii)There exists a k-tree G^* such that $\prod_{1,c}(G^*) < \prod_{1,c}(G)$ and $\prod_2(G^*) > \prod_2(G)$.

Proof of (i)



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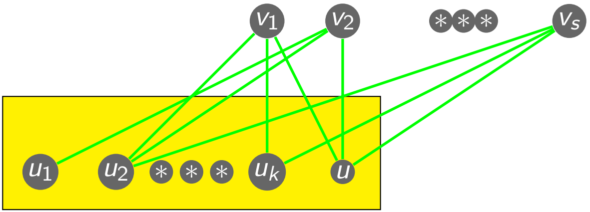
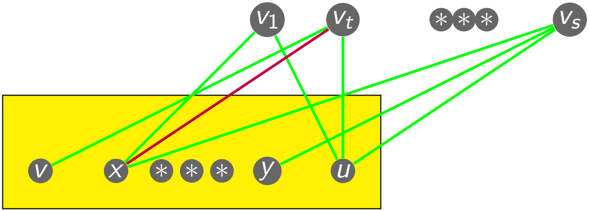
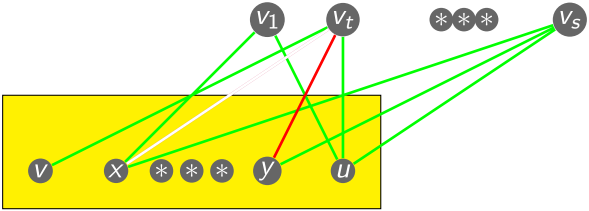


- Let $G' = G[V(G) - \{v_1, v_2 ... v_s\}]$ and $S = N(u) - \{v_1, v_2 ... v_s\}$, we obtain that $d_{G'}(u) = |S| = k$ and $G[S]$ is a k-clique by $u \in S_2$. Since $G \ncong S_n^k$, $d_{G'}(v) \geq k$ for all $v \in S$. And by the facts that $N(v_i) \subseteq (N(u) - \{v_1, v_2 ... v_s\}) \cup \{u\}$ with $|N(v_i)| = k$ and $|(N(u) - \{v_1, v_2 ... v_s\}) \cup \{u\}| = k + 1$, we have that for any $i \in [1, s]$ there exists a vertex $v \in S$ such that $vv_i \notin E(G)$

Proof of (ii)



- Choose v_1, there is a vertex $v \in S$ with $d_{G'}(v) \geq k$. If $d_{G'}(v) = k$, G' is a $k+1$-clique.
- Let $x \in S$ be the vertex such that $d(x) = min_{v \in S}\{d(v)\}$, and let $v_t x \in E(G)$, $v_t y \notin E(G)$ for some $t \in [1, s]$ and $y \in S$, that is, $d(x) - 1 < d(y)$. Denote $G_0 = G[V(G) - \{x, y\}]$.
- Construct a new graph G^* such that $V(G^*) = V(G)$, and $E(G^*) = E(G) - \{v_t x\} + \{v_t y\}$.



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Use the tool

- The function $f(x)=\dfrac{x}{x+m}$ is strictly increasing for $x\in[0,\infty)$, where m is a positive integer.

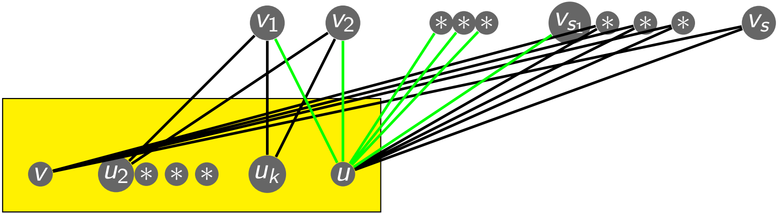
$$\begin{aligned}
\frac{\prod_{1,c}(G)}{\prod_{1,c}(G^*)} &= \frac{[\prod_{w\in V(G_0)} d(w)^c]d(y)^c d(x)^c}{[\prod_{w\in V(G_0)} d(w)^c][d(y)+1]^c[d(x)-1]^c}\\
&= \frac{d(y)^c d(x)^c}{[d(y)+1]^c[d(x)-1]^c}\\
&= \frac{[\frac{d(y)^c}{[d(y)+1]^c}]}{[\frac{[d(x)-1]^c}{d(x)^c}]}\\
&> 1.
\end{aligned}$$

Use the tool

- The function $g(x) = \dfrac{x^x}{(x+m)^{x+m}}$ is strictly decreasing for $x \in [0, \infty)$, where m is a positive integer.

$$
\begin{aligned}
\frac{\prod_2(G)}{\prod_2(G^*)} &= \frac{[\prod_{w \in V(G_0)} d(w)^{d(w)}] d(y)^{d(y)} d(x)^{d(x)}}{[\prod_{w \in V(G_0)} d(w)^{d(w)}][d(y)+1]^{d(y)+1}[d(x)-1]^{d(x)-}} \\
&= \frac{d(y)^{d(y)} d(x)^{d(x)}}{[d(y)+1]^{d(y)+1}[d(x)-1]^{d(x)-1}} \\
&= \frac{\dfrac{d(y)^{d(y)}}{[d(y)+1]^{d(y)+1}}}{\dfrac{[d(x)-1]^{d(x)-1}}{d(x)^{d(x)}}} \\
&< 1.
\end{aligned}
$$

Proof of (ii)



- If $d_{G'}(v) \geq k+1$, reorder the subindices of $\{v_1, v_2 \ldots v_s\}$ so that $vv_i \notin E(G)$ with $i \in [1, s_1]$, where $s_1 \leq s$.
- Construct a new graph G^* such that $V(G^*) = V(G)$, and $E(G^*) = E(G) - \{uv_i\} + \{vv_i\}$, for all $i \in [1, s_1]$
- Since $d(v) \geq k + s - s_1 + 1$ and $d(u) = k + s$, then $d(v) \geq d(u) - s_1 + 1$.

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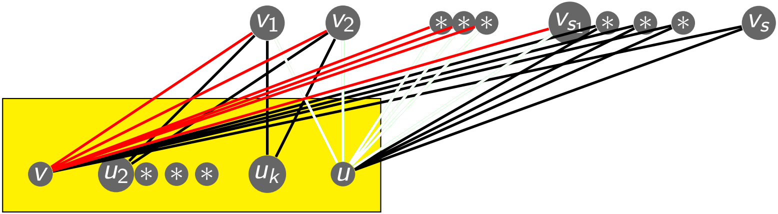

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Use the tool

$$\frac{\prod_{1,c}(G)}{\prod_{1,c}(G^*)} = \frac{d(v)^c d(u)^c}{[d(v)+s_1]^c[d(u)-s_1]^c} = \frac{[\frac{d(v)^c}{[d(v)+s_1]^c}]}{[\frac{[d(u)-s_1]^c}{d(u)^c}]} > 1.$$

$$\frac{\prod_2(G)}{\prod_2(G^*)} = \frac{d(v)^{d(v)} d(u)^{d(u)}}{[d(v)+s_1]^{d(v)+s_1}[d(u)-s_1]^{d(u)-s_1}} = \frac{[\frac{d(v)^{d(v)}}{[d(v)+s_1]^{d(v)+s_1}}]}{[\frac{[d(u)-s_1]^{d(u)-s_1}}{d(u)^{d(u)}}]} < 1.$$

Sketch of the Proof

The proof of $\prod_{1,c}(T_n^k) \leq \prod_{1,c}(P_n^k)$ and $\prod_2(T_n^k) \geq \prod_2(P_n^k)$:

Sketch of the Proof

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- Let G be a k-tree, assume that either $\prod_{1,c}(G)$ attains the maximum or $\prod_2(G)$ attains the minimum.

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- By contradiction, we can show that every hyper pendent edge is a k-path.
- By induction, we can prove that $|S_1(G)| = 2$, thus, P_n^k attains the maximal $\prod_{1,c}(G)$ and minimal $\prod_2(G)$.

Thank you