Inductive tools for handling internally 4-connected binary matroids and graphs

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Mississippi Discrete Mathematics Workshop, November 2013

Matroids from graphs and matrices

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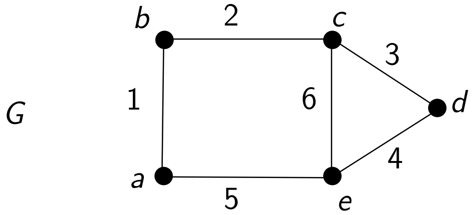

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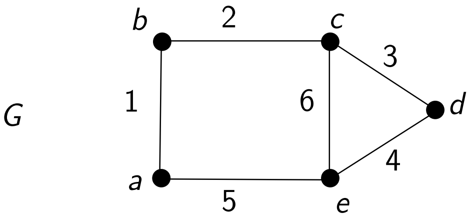

The vertex-edge incidence matrix of G (over the 2-element field):

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Matroids from graphs and matrices

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}$$
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$\mathcal{C}$ (the circuits): minimal linearly dependent sets of columns

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We get a binary matroid $M[A]$ from any matrix A over the 2-element field.

Every graphic matroid is binary.

2-connected matroids and graphs

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Such matroids are also called connected.

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Let G be a graph with at least three vertices and without isolated vertices.

$M(G)$ is 2-connected if and only if G is 2-connected and loopless.

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Theorem (Tutte, 1966)

Let M be a 2-connected matroid and e be an element of M. Then $M\backslash e$ or M/e is 2-connected.

3-connected graphs and matroids

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Let G be a graph with at least four vertices and without isolated vertices.

$M(G)$ is 3-connected if and only if G is 3-connected and simple.

3-connected graphs and matroids

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Tutte's Wheels Theorem (1961)

Let G be a 3-connected simple graph. Then G has a 3-connected simple minor G' with

$$|E(G) - E(G')| = 1 \text{ \textit{unless}}$$

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In the exceptional case, G has a 3-connected simple minor G' with

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For matroids in general, this is not a well-defined operation. But it is well-defined for binary matroids.

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A 3-connected binary matroid is internally 4-connected if it does not break up as a 3-sum.

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For matroids in general, this is not a well-defined operation. But it is well-defined for binary matroids.

A 3-connected binary matroid is internally 4-connected if it does not break up as a 3-sum.

A 3-connected simple graph is internally 4-connected if it is 4-connected except for the possible presence of degree-3 vertices.

Internally 4-connected binary matroids

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A graph is internally 4-connected if and only if its cycle matroid is internally 4-connected.

Internally 4-connected binary matroids

Theorem (2011)

Let M be an internally 4-connected binary matroid. Then M has an internally 4-connected minor M' with

$$1 \leq |E(M) - E(M')| \leq 3$$

unless

Internally 4-connected binary matroids

Theorem (2011)

Let M be an internally 4-connected binary matroid. Then M has an internally 4-connected minor M' with

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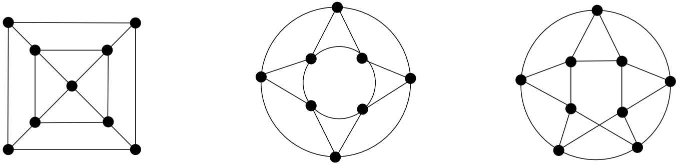
unless M or M^* is the cycle matroid of

(i) a terrahawk; or (ii) a planar or Möbius quartic ladder.



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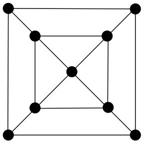

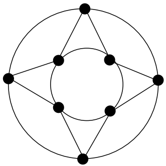

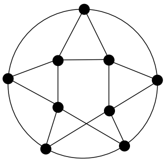

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Let M be a 3-connected matroid and N be a 3-connected proper minor of M.

Goal. To remove a small number of elements from M and keep

(i) 3-connectedness; and
(ii) an isomorphic copy of N.

Seymour's Splitter Theorem

Seymour's Splitter Theorem (1980)

Let M be a 3-connected matroid and N be a 3-connected proper minor of M. Then M has a 3-connected minor M' that has an N-minor such that

$$|E(M) - E(M')| = 1$$

unless M is a wheel or a whirl.

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In the exceptional case, M has a 3-connected minor M' with an N-minor such that

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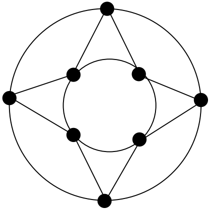

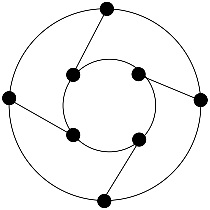

A Splitter Theorem for internally 4-conn. binary matroids

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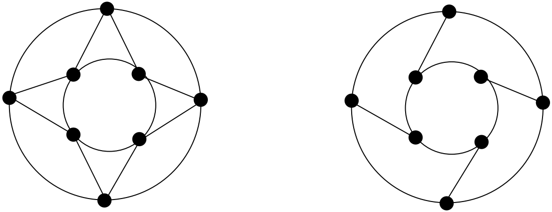

Consequence. We cannot remove some bounded set of elements to recover internal 4-connectivity.

Earlier work

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For graphs: Johnson and Thomas (2002)

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For binary matroids: Geelen and Zhou (2006), Zhou (2012)

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All allow the intermediate matroid to satisfy some weaker form of connectivity.

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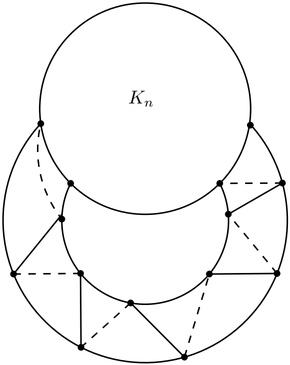
Alternative approach. (Geelen) Expand the types of moves one allows to go from M to M'.

Another move

Delete all the dashed edges taking a quartic ladder segment to a cubic ladder segment.



3-separations

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A 3-separation in a matroid M is a partition (X, Y) such that

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and

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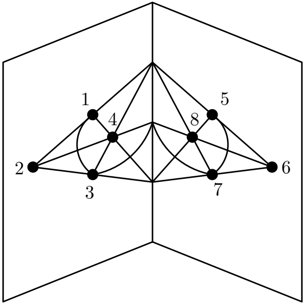
Example. $(\{1,2,3,4\}, \{5,6,7,8\})$ is a 3-separation of the binary affine space $AG(3,2)$.



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Other variants on 4-connectivity allow certain restricted kinds of 3-separations.

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Always

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Suppose M is 4-connected.

M is an internally 4-connected matroid with no triangles and no triads.

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Theorem (2012)

When M is 4-connected, it has an internally 4-connected minor M' that has an N-minor such that

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unless M is a certain 16-element self-dual matroid. In the exceptional case, there is a 2-element WIN.

The 16-element exception

Let D_{16} be the 16-element rank-8 matroid that is represented over $GF(2)$ by the matrix $[I_8|A]$ where A is

$$\left(\begin{array}{cccc|cccc}
1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
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-- & -- & -- & -- & -- & -- & -- & -- \\
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Evidently D_{16} is isomorphic to its dual. Moreover, D_{16} has two visible $AG(3,2)$-minors on disjoint ground sets.

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Can we say something about the connectivity of $M \backslash e$?

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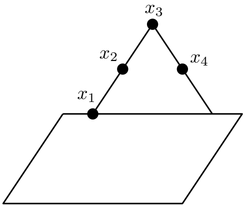
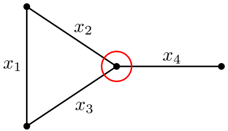
A 3-connected binary matroid is $(4, 4, S)$-connected if one side of every 3-separation is a triangle, a triad, or a 4-element fan.





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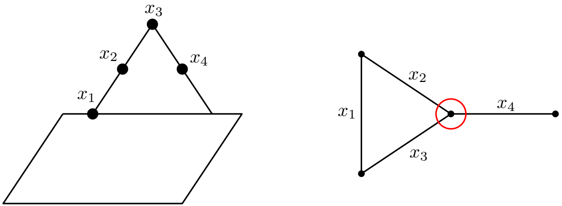

Step 2 gave us a triangle containing an element e such that $M\backslash e$ has an N-minor.

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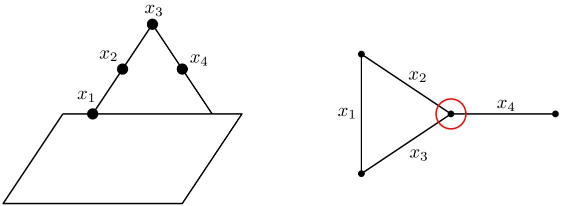

Step 2 gave us a triangle containing an element e such that $M\backslash e$ has an N-minor.

We want to say something about the connectivity of $M\backslash e$.

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Theorem

Suppose $|E(M)| \geq 15$. *Then*

(i) *there is a* 1-, 2-, *or* 3-*element win; or*

(ii) M *or* M^* *is a cubic Möbius or planar ladder or a special single-element coextension thereof; or*

(iii) *up to duality,* M *has a triangle* T *and an element* e *such that* $M \backslash e$ *has an* N-*minor and is* $(4, 4, S)$-*connected.*

Step 3

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A $(4, 4, S)$-connected matroid has one side of every 3-separation as a triangle, a triad, or a 4-fan.

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(iii) *up to duality, M has a triangle T and an element e such that $M \backslash e$ has an N-minor and is $(4, 4, S)$-connected.*

What happens if (iii) of the last theorem holds?

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Assume M has a triangle T and an element e such that
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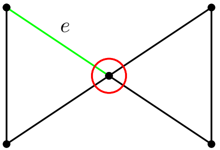

Figure : A good bowtie

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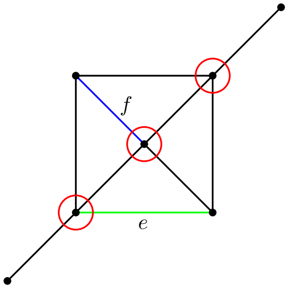

Figure : An augmented 4-wheel: $M \backslash f$ has an N-minor.

Step 4

Theorem

Suppose $|E(M)| \geq 16$ and M has a triangle T and an element e such that $M \backslash e$ has an N-minor and is $(4, 4, S)$-connected. Then

(i) *there is a 1-, 2-, 3-, or 4-element win; or*

(ii) *M or M^* has a good bowtie; or*

(iii) *M or M^* has an augmented 4-wheel.*

Step 4

Theorem

Suppose $|E(M)| \geq 16$ *and* M *has a triangle* T *and an element* e *such that* $M\backslash e$ *has an* N*-minor and is* $(4, 4, S)$*-connected. Then*

(i) *there is a* 1-, 2-, 3-, *or* 4-*element win; or*

(ii) M *or* M^* *has a good bowtie; or*

(iii) M *or* M^* *has an augmented* 4-*wheel.*

Can we eliminate (iii)?

Step 5: Killing augmented 4-wheels

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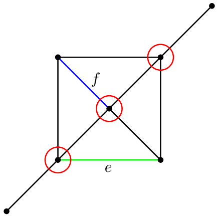


- $M \backslash e$ is $(4, 4, S)$-connected having an N-minor; and
- $M \backslash f$ has an N-minor.

Step 5: Killing augmented 4-wheels

Theorem

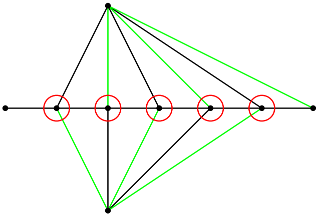
Suppose $|E(M)| \geq 16$ and M has an augmented 4-wheel. Then

(i) *there is a 1-, 2-, 3-, or 4-element win; or*

(ii) *M has a good bowtie; or*

(iii) *M is a terrahawk; or*

(iv) *M contains the configuration shown below where the deletion of all of the green elements is an internally 4-connected matroid having an N-minor.*



Step 5: Killing augmented 4-wheels

Theorem

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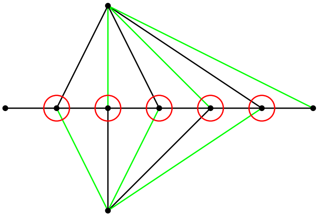

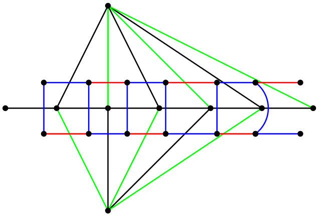

When bowties are good

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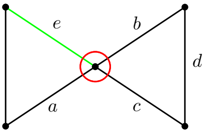

Let M have a good bowtie as shown, so $M\backslash e$ is $(4, 4, S)$-connected having an N-minor.

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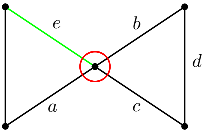
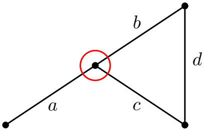


Let M have a good bowtie as shown, so $M\backslash e$ is $(4, 4, S)$-connected having an N-minor.

Then $M\backslash e$ has a 4-fan.



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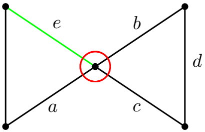

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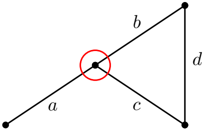

As N is internally 4-connected, it contains no 4-fans.

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Lemma

$M\backslash e\backslash d$ or $M\backslash e/a$ has an N-minor.

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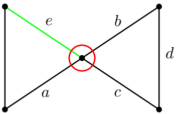

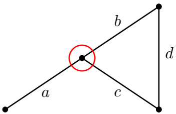

Lemma

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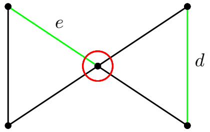
Lemma

If M has a good bowtie, then it has a good bowtie in which $M\backslash e\backslash d$ has an N-minor.

Building from a good bowtie



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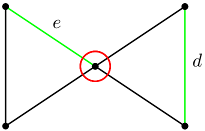
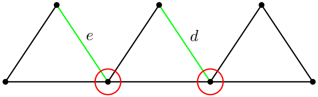


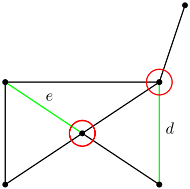
Theorem

Suppose M has a good bowtie in which $M\backslash e\backslash d$ has an N-minor. Then

(i) *$M\backslash d$ is $(4,4,S)$-connected and M contains one of the two structures shown; or*

(ii) *$M\backslash d$ is not $(4,4,S)$-connected.*





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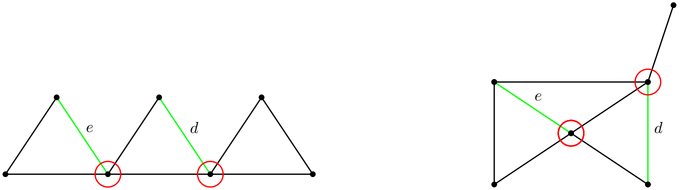
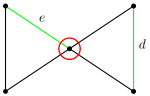


Case (ii) gives a WIN.

Building from a good bowtie



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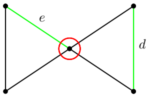
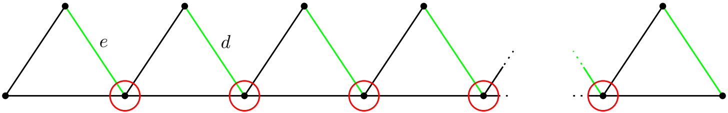


Theorem

When M has a good bowtie in which $M\backslash e\backslash d$ has an N-minor but $M\backslash d$ is not $(4, 4, S)$-connected, one of the following occurs:

(a) *there is a 1-, 2-, 3-, or 4-element win; or*

(b) *M contains the configuration shown and deleting all of the green elements gives an internally 4-connected matroid having an N-minor.*



Chains of bowties

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What other moves do we need?

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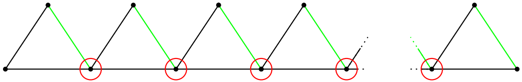
Bowtie moves:

Chains of bowties

Bowtie moves: We've seen chains of bowties.



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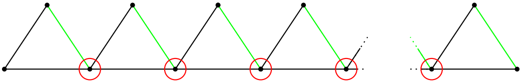
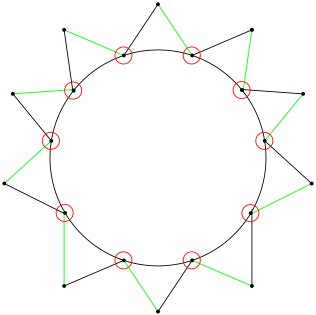


We can also have rings of bowties.



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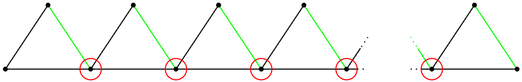

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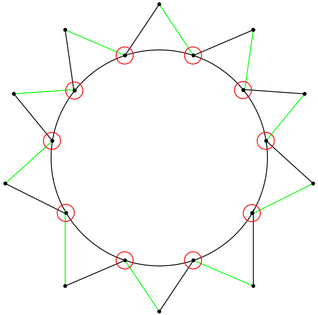

Only expect small variants on these moves.

The endgame

The endgame

What's left to do?

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What's left to do? Handling chains of bowties.

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Steps 6, 7, and 8

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Consequence: *A new theorem for internally 4-connected graphs.*

Summary

Two powerful inductive tools for 3-connected matroids and graphs.

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Corresponding inductive tools for internally 4-connected binary matroids and graphs.

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(1) By removing at most six elements from an internally 4-connected binary matroid, we can recover internal 4-connectivity.

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(1) By removing at most six elements from an internally 4-connected binary matroid, we can recover internal 4-connectivity.

(2) By removing elements from an internally 4-connected binary matroid using a small number of well-described moves, we can keep internal 4-conn. and a copy of an internally 4-conn. minor.